Yang-Mills instantons over Hopf surfaces

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

The 4-manifold $S^1 \times S^3$, when endowed with the structure of a certain complex Hopf surface, is an example of a principal elliptic fibration. We use this structure to study the moduli spaces of anti-self-dual connections (instantons) on SU(2) bundles over $S^1 \times S^3$.

Chapter 1 is introductory. We define Buchdahl's notion of stability and outline the correspondence between instantons and stable holomorphic SL(2, $\mathbf{C}$) bundles over $S^1 \times S^3$. In Chapter 2 we study holomorphic line and SL(2, $\mathbf{C}$) bundles over a general principal elliptic surface using an extension of the 'graph' invariant introduced by Braam and Hurtubise. We prove some auxiliary results needed in later chapters and introduce a stratification of the moduli space.

In Chapter 3 we construct elements of one of the strata using the 'Serre construction' of algebraic geometry and deduce a structure result for the charge 1 case.

Chapter 4 applies the results of the previous chapters in the construction of monopoles on the solid torus with a hyperbolic metric. We recover easily a result of Braam and Hurtubise.

In Chapter 5 we adapt a construction of Friedman to describe a method of construction for elements of the remaining strata of the moduli spaces over the Hopf surface. In the charge 1 case we again determine the diffeomorphism type of the stratum completely. Combined with the results of Chapter 3 we deduce the natural action of $S^1 \times S^3$ on the charge 1 moduli space is free.

In Chapter 6 we study the charge 1 instanton moduli spaces over secondary Hopf surfaces diffeomorphic to the product of S^1 and a Lens space.

Chapter 7 considers twistorial methods and their application in the construction of explicit solutions. We define an invariant of an instanton, the spectral surface, which is a 2-dimensional analogue of Hitchin's spectral curve. We use it to deduce that methods of Atiyah and Ward fail to generate a full family of charge 1 solutions. Finally we show how the spectral surface can be used in a sheaf theoretic construction of the 'missing' solutions.

Notation and conventions

If X is a complex manifold we denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X, and by $\mathcal{K}_X$ the canonical bundle of X. We denote the Picard group $H^1(X;\mathcal{O}_X^*)$ of isomorphism classes of holomorphic line bundles on X by $\mathrm{Pic}(X)$. The subgroup consisting of classes whose first Chern class vanishes is denoted by $\mathrm{Pic}^0(X)$. We shall denote objects on X which have holomorphic structures by script letters, and objects with smooth structures by roman letters. Thus a holomorphic vector bundle may be denoted by $\mathcal{E}$, whereas the underlying smooth bundle will be denoted by E. We shall normally drop the words 'holomorphic' and 'smooth' when the structure is clear from the context. For given integers p and q, we use $\Omega^p(E)$ to denote the smooth p-forms on X with values in E, and $\Omega^{p,q}(E)$ the smooth E-valued forms of bidegree (p,q).

There is a natural isomorphism between the category of holomorphic vector bundles over X and the category of locally free sheaves of $\mathcal{O}_X$-modules given by associating to a holomorphic vector bundle $\mathcal{E}$ its sheaf of holomorphic sections $\mathcal{O}(\mathcal{E})$. We shall usually make this identification. However, since we shall also denote coherent sheaves of $\mathcal{O}_X$-modules by script letters, this may on occasion lead to confusion. In such instances we shall always state whether mean 'sheaf' or 'vector bundle'.

We denote by $\mathcal{E}xt_{\mathcal{O}_X}$ the sheafification of the functor Ext from the category of $\mathcal{O}_X$-modules to the category of Abelian groups. Likewise we denote the sheafification of Hom by $\mathcal{H}om_{\mathcal{O}_X}$. Given coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we shall usually denote the tensor product $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ simply by $\mathcal{F}\mathcal{G}$.

We shall sometimes use the notation $\mathcal{O}_X(L)$ for the holomorphic line bundle defined by a divisor L. We use the notation $L_1 \sim L_2$ to indicate when two divisors L_1, L_2 on X are linearly equivalent. For a given point $x \in X$ the local intersection number of L_1 and L_2 at x is denoted $L_1 ._x L_2$. The global intersection number is denoted by $L_1 . L_2$.

By a subscheme $(Z,\mathcal{O}_Z)$ of X we mean the zero-set defined by a sheaf of ideals $\mathcal{J}_Z \subset \mathcal{O}_X$ together with its structure sheaf $\mathcal{O}_Z$ defined to be the quotient $\mathcal{O}_X/\mathcal{J}_Z$. Usually we shall omit $\mathcal{O}_Z$ from the notation and refer simply to 'a subscheme Z'. The integer $\dim_{\mathbf{C}} H^0(\mathcal{O}_Z)$ is denoted by $l(Z)$. Likewise, a *skyscraper sheaf* $\mathcal{S}$ on X is a coherent sheaf supported on a finite number of points and we define $l(\mathcal{S}) = \dim_{\mathbf{C}} H^0(\mathcal{S})$.

Finally, when M is a closed oriented n-dimensional (real) manifold, the orientation defines an isomorphism $H^n(M;\mathbf{Z}) \cong \mathbf{Z}$, and we shall normally identify the two groups. In particular, when M is 4-dimensional and $E \to M$ is a smooth complex vector bundle, the characteristic classes $c_1(E)^2$ and $c_2(E)$ are always taken to be integers. If M is a complex surface, we use the natural orientation induced by the complex structure.

Introduction

The original motivation for this thesis stemmed from the introduction by Floer [22] of a homology theory constructed from the moduli spaces of finite action anti-self-dual Yang-Mills instantons over $\mathbf{R} \times Y$, where Y is a homology 3-sphere. It was a natural question to ask whether one could deduce similar or other invariants of Y from the instanton moduli spaces over $S^1 \times Y$. It soon became apparent, however, that although there was a great deal of literature on the instanton moduli spaces associated to simply-connected 4-manifolds, relatively little was known about the corresponding spaces over 4-manifolds with non-trivial fundamental group, and the question of the structure of these spaces became more interesting in its own right.

The first advances in this direction were made by Braam and Hurtubise [16], who studied SU(2) instantons over the simplest example $S^1 \times S^3$. In particular they were able to determine the structure of the moduli space $\mathcal{M}_1$ of charge 1 instantons. Using the natural action of the subgroup $S^1 \times S^1$ of the isometry group of $S^1 \times S^3$ on the moduli space, they proved that $\mathcal{M}_1$ is a principal $S^1 \times S^1$-bundle over $S^2 \times \mathbf{R}^4$ with total space diffeomorphic to $S^1 \times S^3 \times \mathbf{R}^4$.

Earlier work of Braam [14, 15] identified monopoles on the solid torus $D^2 \times S^1$, where D^2 is the 2-disc with its standard hyperbolic metric, with instantons on $S^1 \times S^3$ invariant under an action of S^1. This led to the second main result of [16]; namely that the moduli space of monopoles on $D^2 \times S^1$ of mass m and charge 1 is diffeomorphic to $D^2 \times S^1 \times S^1$.

The critical observation of Braam and Hurtubise is that $S^1 \times S^3$ can be given the structure of a complex principal elliptic surface X: it is an example of a Hopf surface. A theorem of Buchdahl [18] then gives a diffeomorphism between the instanton moduli spaces and the moduli spaces of stable holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles over X, changing the problem to one of complex analysis. To each isomorphism class of holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundle over X, Braam and Hurtubise associate a divisor in $\mathbf{P}^1 \times \mathbf{P}^1$. Their analysis of the moduli space then proceeds in an essentially constructive manner. Given a suitable divisor, they construct $\mathrm{SL}(2, \mathbf{C})$ bundles locally with the same divisor and patch together. The hard work involved in their proof lies in showing that this gluing can actually be done, when it results in a stable bundle and finally how changing the 'glue' affects the isomorphism class of the resulting bundle.

Around the same time as the work of Braam and Hurtubise, algebraic geometers were studying stable rank 2 algebraic bundles over *simply-connected algebraic* elliptic surfaces. Following Donaldson's use of a specific Dolgachev surface in his counterexample to the h-cobordism conjecture in four dimensions [20], the moduli space of stable holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles over more general Dolgachev surfaces was studied extensively, notably by Friedman and Morgan [25], and Okonek and van de Ven [40]. More generally still, in his paper [24] Friedman constructed stable holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles over arbitrary simply-connected algebraic surfaces using an invariant similar to that of [16]. The constructions of the aforementioned authors involve extensive use of coherent sheaves and their associated algebra. In particular they are of an intrinsically global nature.

Although $S^1 \times S^3$ is neither algebraic nor simply-connected, the algebraic methods mentioned above, and particularly those of Friedman, are quite general and can be applied also in this case. It is the aim of the first part of this thesis to show how these methods can be adapted to the case of $S^1 \times S^3$. After setting up the necessary machinery, we find the results of Braam and Hurtubise follow quite easily and can even be strengthened. Indeed the group $S^1 \times \mathrm{SU}(2)$ acts on $S^1 \times S^3$ by isometries, extending the action of $S^1 \times S^1$. We shall prove that the corresponding action on $\mathcal{M}_1$ is free. This result corresponds to the intuitive idea that each instanton has an associated 'centre'. In the charge 1 case at least, these constructions are somewhat simpler than those of Braam and Hurtubise in that they reduce to pure algebra. Although they do involve a fair amount of algebraic machinery, the rôle played by configurations of points ('centres') becomes somewhat clearer.

The thesis is organised as follows. In Chapter 1 we outline the results of Buchdahl and Braam and Hurtubise relating instantons and stable holomorphic bundles, principally to fix notation. In Chapter 2 we place the 'graph' or 'divisor' invariants for holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles used by Braam, Hurtubise and Friedman in the more general context of principal elliptic fibrations and prove some of the auxiliary results required in later chapters. In particular we consider the obstruction to the existence of certain line bundles over principal elliptic fibrations and questions of stability. Finally we observe that the divisors provide a means of stratifying the moduli spaces.

The 'lowest stratum' consists of bundles which can be constructed by sheaf extensions using a result of Serre. This is the method applied to Dolgachev surfaces in [25] and [40]. In Chapter 3 we review the corresponding construction for $S^1 \times S^3$ and determine when it leads to stable bundles. Since the Picard group of the Hopf surface is not discrete, the stability condition is slightly more complicated. The treatment we give combines aspects of both [16] and [24]. In the charge 1 case the construction becomes particularly simple and the stratum is a complex 3-manifold fibred by Hopf surfaces.

Having set up all the machinery of commutative algebra in Chapter 3, in Chapter 4 we digress to monopoles on $D^2 \times S^1$ and find that the theorem of

Braam and Hurtubise is obtained with little extra effort. An application of the residue theorem determines stability.

In Chapter 5 we return to the case of instantons over the Hopf surface and describe a construction for those stable bundles which cannot be obtained by the method of Chapter 3. The method is that used by Friedman in [24] for simply connected algebraic surfaces. Over a double cover of the Hopf surface X we construct a certain line bundle and push it down to obtain a rank 2 bundle on X. The required $\mathrm{SL}(2,\mathbf{C})$ bundle is then given as an extension of a certain torsion sheaf by this rank 2 bundle. Many of Friedman's results can also be applied to the Hopf surface, and for those which cannot, because they are specific to the algebraic case, we give our own. In particular, the existence of the required line bundle follows by a result of Chapter 2. The method used by Friedman to determine the generic stratum for the most part goes through, but we need a little extra argument to exclude 'borderline' cases. Finally, we use the construction to determine the structure of the generic stratum of the moduli space of charge 1 instantons. Knowledge of the structure of the strata then allows us to deduce information about the action of $S^1 \times \mathrm{SU}(2)$.

In Chapter 6 we observe that the product of S^1 with certain Lens spaces is also a principal elliptic fibration. In two special cases the results of the previous chapters allow us to determine the diffeomorphism type of the corresponding charge 1 instanton moduli spaces including the result of Braam and Hurtubise for the case of $S^1 \times S^3$. The only essential difference is the appearance of torsion in the fundamental group and its consequences for stability. We have to remove a set of 'bad points' which gives rise to 'new' ends of the moduli space. This is a result expected from representation theory.

Chapter 7 forms the second part of the thesis. In it we address the problem of finding explicit solutions to the anti-self-duality equations over $S^1 \times S^3$ in the form of connections. A 3-dimensional family has already been found by Braam [14]. The principal tools are twistor spaces and the Ward correspondence, which identifies $\mathrm{SU}(2)$ instantons on $S^1 \times S^3$ with certain holomorphic $\mathrm{SL}(2,\mathbf{C})$ bundles on its twistor space Z. The twistor space Z is an elliptic 3-fold and the general results of Chapter 2 associate a divisor invariant to each instanton bundle on Z. Using ideas of Garland and Murray [27, 28] first applied to the case of instantons on $S^1 \times \mathbf{R}^3$, we identify $\mathrm{SU}(2)$ instantons on $S^1 \times S^3$ with monopoles on a bundle over S^3 with gauge group a semi-direct product of S^1 with the loop group of $\mathrm{SU}(2)$. The divisor invariant can then be viewed as a *spectral surface* for the corresponding monopole, provided one replaces the asymptotic decay conditions of Hitchin's original definition for spaces whose geodesics are non-compact [33] with a monodromy condition around the closed geodesics in S^3.

Unfortunately, the spectral surface of a general instanton appears to be a rather complicated object and we quickly return to the charge 1 case, where we can identify it relatively easily. We then use the spectral surface to show that, unlike the case of S^4, on $S^1 \times S^3$ the constructions of 't Hooft, Atiyah and Ward [9] fail to fill out the whole moduli space, reflecting the dearth of curves in the twistor space and the absence of an ample line bundle.

The main result of this chapter is a construction of a generic 1-instanton. This combines a 3-dimensional analogue of the Friedman construction and blow-downs. Crucial in this part is the knowledge of the structure of the spectral surface which allows us to construct the necessary line bundles. Disappointingly, it is not clear how to obtain the corresponding connections using this construction, but it may prove useful in the study of other properties of the moduli space such as the action of the isometry group of $S^1 \times S^3$.

Chapter 1

Preliminaries

In this chapter we recall briefly the anti-self-duality equations for connections on an SU(2) bundle over an oriented Riemannian 4-manifold M, and describe the relationship between anti-self-dual connections and holomorphic vector bundles when M is a complex Hermitian surface. In so doing we shall introduce the notation and fundamental results which we shall use throughout, and in particular indicate how these apply to Hopf surfaces. These results are for the most part standard and this chapter contains no original material. The main background references on the Yang-Mills equations are [21], [23] and [35]. For the material on complex geometry see [16], [18], [21], [30] and [34].

1.1. Geometry of $S^1 \times S^3$

In this section we show how $S^1 \times S^3$ may be regarded as a complex manifold, and describe some of its complex analytic geometry.

Fix $\lambda \in \mathbf{R}^*$ with $|\lambda| > 1$. There is a free action of $\mathbf{Z}$ on $\mathbf{R}^4 \setminus \{0\}$ generated by the map

1.1.1. $$x \mapsto \lambda x,$$

with quotient space $(\mathbf{R}^4 \setminus \{0\})/\mathbf{Z}$ diffeomorphic to $S^1 \times S^3 = (\mathbf{R}/2\pi\mathbf{Z}) \times S^3$ via the map defined by

$$x \mapsto (2\pi \log|x| / \log\lambda, x/|x|).$$

We identify $\mathbf{R}^4 \setminus \{0\}$ with the non-zero quaternions $\mathbf{H}^*$ via the map

1.1.2. $$(x_0, x_1, x_2, x_3) \mapsto x_0 + x_1 i + x_2 j + x_3 k.$$

Regarding $\mathbf{H}$ as a left $\mathbf{H}$-module, multiplication by an element of

1.1.3. $$\mathbf{S} = \{\sigma \in \mathrm{Im}\mathbf{H} : |\sigma| = 1\},$$

induces a complex structure on $\mathbf{H}^*$ with respect to which the action of $\mathbf{Z}$ defined by (1.1.1) is holomorphic. Hence each element of $\mathbf{S}$ endows $S^1 \times S^3$ with the structure of a complex manifold. (In particular, the quaternions i, j and k define integrable complex structures I, J and K respectively, which satisfy $IJ = -JI = K$, so $S^1 \times S^3$ is a so-called *hyper-complex manifold*). Note that since $H^2(S^1 \times S^3; \mathbf{Z}) = 0$ there can be no Kähler metric on $S^1 \times S^3$. However the standard metric on $S^1 \times S^3$ possesses properties which will prove useful.

The metric $\langle \cdot, \cdot \rangle$ on $\mathbf{H}^*$ given at a point $z \in \mathbf{H}^*$ by

1.1.4. $$\langle \xi, \eta \rangle = \kappa \frac{\mathrm{Re}(\xi \bar{\eta})}{|z|^2},$$

where $\xi, \eta \in T_z(\mathbf{H}^*) \cong \mathbf{H}$ and $\kappa \in \mathbf{R}_{>0}$ is a constant, is invariant under the action of $\mathbf{Z}$ and $\mathbf{S}$ and so descends to give a metric on $S^1 \times S^3$ which is Hermitian with respect to every complex structure in $\mathbf{S}$. Choosing $\kappa = (4\pi^2 \log \lambda)^{-1/2}$ normalises the metric so that $\mathrm{vol}(X) = 1$. The resulting metric is conformally equivalent to the product metric on $S^1 \times S^3$ given by the the usual metric on S^1 (rescaled by a factor of $(\log \lambda)/2\pi$) and the usual metric on S^3. Note that this metric has positive scalar curvature and is conformally flat.

Give $S^1 \times S^3$ the complex structure defined by $i \in \mathbf{S}$ and denote the resulting Hopf surface by X. If we identify $\mathbf{C}^2$ with $\mathbf{H}$ via the map

$$(z_0, z_1) \longmapsto z_0 + z_1 j,$$

then the metric (1.1.4) is given by

$$\kappa\left(\frac{dz_0 \otimes d\bar{z}_0 + dz_1 \otimes d\bar{z}_1}{|z_0|^2 + |z_1|^2}\right),$$

and the associated Hermitian (1,1) form ω by

$$\omega = \frac{i\kappa}{2}\left(\frac{dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1}{|z_0|^2 + |z_1|^2}\right).$$

It is then easy to check that ω satisfies

$$\bar{\partial}\partial\omega = 0.$$

The non-zero complex numbers $\mathbf{C}^*$ act freely on the left of $\mathbf{C}^2 \setminus \{0\}$ by scalar multiplication, extending the action of $\mathbf{Z}$ defined by (1.1.1). Consequently X inherits a free holomorphic $\mathbf{C}^*/\mathbf{Z} = T$ action giving the quotient map

1.1.5. $$\pi : X \longrightarrow \mathbf{P}^1 \cong (\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^*$$

the structure of a principal elliptic fibration [12]. Here T is the elliptic curve $\mathbf{C}^*/\mathbf{Z}$ given by the quotient of $\mathbf{C}^*$ by the $\mathbf{Z}$-action generated by the map $z \mapsto \lambda z$.

The group $\mathrm{GL}(2, \mathbf{C})$ acts on X by complex automorphisms and with stabiliser subgroup $\{\lambda^n I : n \in \mathbf{Z}\} \cong \mathbf{Z}$, identifying the quotient group $\mathrm{GL}(2, \mathbf{C})/\mathbf{Z}$

with $\mathrm{Aut}_0(X)$, the identity component of the automorphism group of X. Using the description of X in terms of quaternions, the natural left and right actions of $\mathbf{H}^*$ on itself induce corresponding actions on X. The right action of $\mathbf{H}^*$ is holomorphic, preserves the metric and has stabiliser subgroup $\mathbf{Z}$ generated by λ. This then defines a free right action of X on itself by complex automorphisms, the action of $w_0 + w_1 j \in X$ corresponding to the left action of the matrix

$$\begin{pmatrix} w_0 & -\overline{w}_1 \\ w_1 & \overline{w}_0 \end{pmatrix} \in \mathrm{GL}(2,\mathbf{C})/\mathbf{Z}.$$

We make this into a left action in the usual way by defining the left action of $g \in X$ as

$$g.x = x.g^{-1}, \text{ for } x \in X.$$

The subgroup $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ of $\mathbf{H}^*$ acts freely on the left preserving the metric, but 'rotates' the complex structures. Denoting these two groups by $S^1 \times \mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ respectively, we have a left action of $S^1 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ on X, covering that of the isometry group $S^1 \times \mathrm{SO}(4)$. The factor $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ corresponds to the double cover $\mathrm{Spin}(4)$ of $\mathrm{SO}(4)$.

1.2. The Anti-Self-Duality Equations

Let M be a closed oriented Riemannian 4-manifold. The metric and orientation on M define a Hodge $*$-operator

$$* : \Omega^p(M) \to \Omega^{4-p}(M),$$

where $\Omega^p(M) = \Gamma(\Lambda^p T^* M)$. Setting $p = 2$, we have a linear map

$$* : \Omega^2(M) \to \Omega^2(M),$$

such that $*^2 = 1$, and a corresponding decomposition of $\Omega^2(M)$ into ± 1 eigenspaces:

1.2.1. $$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M).$$

where $\Omega^2_+(M)$ and $\Omega^2_-(M)$ are the *self-dual* and *anti-self-dual* two forms respectively. Clearly we obtain an analogous decomposition of $\Omega^2(E)$ for any smooth vector bundle E over M.

Let $E \to M$ be an $\mathrm{SU}(2)$ bundle; i.e., a rank 2 complex vector bundle with a Hermitian metric on its fibres and with $\Lambda^2 E$ trivialised. It has one topological invariant: its second Chern class $c_2(E) \in H^4(M;\mathbf{Z}) \cong \mathbf{Z}$. Let $\mathcal{A}$ denote the space of connections on E which preserve the $\mathrm{SU}(2)$ structure, i.e., both the metric and trivialisation of $\Lambda^2 E$. (Sometimes we shall refer simply to "a connection on E", when the structure is understood). Given a connection $\nabla \in \mathcal{A}$, denote its curvature by F_∇. Note that F_∇ is an element of $\Omega^2(\mathrm{End}(E))$. Denote by $\mathcal{G}$ the group of complex linear automorphisms of E

which preserve the metric and trivialisation. The group $\mathcal{G}$ is called the *group of gauge transformations* and acts on $\mathcal{A}$ as follows. Given $g \in \mathcal{G}$ and $\nabla \in \mathcal{A}$, we define

$$g.\nabla = g \circ \nabla \circ g^{-1} : \Omega^0(E) \to \Omega^1(E).$$

The curvatures of $g.\nabla$ and ∇ are related by

$$F_{g.\nabla} = g \circ F_\nabla \circ g^{-1} \in \Omega^2(\mathrm{End}(E)).$$

Suppose $c_2(E) = k > 0$. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ consisting of irreducible connections ∇ which satisfy the *anti-self-duality equation*:

1.2.2. $$F_\nabla = - * F_\nabla.$$

If a connection ∇ satisfies equation 1.2.2 we say ∇ is *anti-self-dual*, and sometimes call it a *k-instanton* or just *instanton*. The integer k is called the *charge* of the instanton. Instantons are the absolute minima of the *Yang-Mills functional* $\mathcal{YM}$ defined on $\mathcal{A}$ by

1.2.3. $$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M |F_\nabla|^2 \mathrm{vol}_M.$$

If a connection ∇ satisfies equation 1.2.2 then so does $g.\nabla$ for all $g \in \mathcal{G}$, corresponding to the invariance of $\mathcal{YM}$ under the action of $\mathcal{G}$. The group $\mathcal{G}$ therefore acts on $\mathcal{B}$. We shall study the *moduli space* $\mathcal{M}_k = \mathcal{B}/\mathcal{G}$ of irreducible anti-self-dual connections on E.

The following theorem is due to Atiyah, Hitchin and Singer [8], and Taubes [42].

Theorem 1.2.4. *Suppose M is a closed oriented anti-self-dual Riemannian 4-manifold and let $E \to M$ be an $\mathrm{SU}(2)$ bundle with $c_2(E) > 0$. Then the moduli space $\mathcal{M}_k$ of irreducible anti-self-dual connections on E is a non-empty smooth $8k - 3(1 - b_1 + b_2^+)$ dimensional manifold, where b_1 is the first Betti number of M and b_2^+ is the dimension of the space $H_+^2(M)$ of self-dual harmonic 2-forms.*

The functional (1.2.3) is invariant not only under the action of the group of isometries of M, but also under the action of the group of conformal transformations. The identity components of these groups therefore act on $\mathcal{M}_k$: the action of an element φ of either group lifts to an action on E and we can define $\varphi.\nabla = \varphi \circ \nabla \circ \varphi^{-1}$. The lift is defined uniquely up to the action of $\mathcal{G}$, hence we obtain a well-defined action on the moduli spaces $\mathcal{M}_k$.

1.3. Holomorphic vector bundles on complex surfaces

Let X be a complex surface. A holomorphic vector bundle $\mathcal{E}$ on X carries a natural Dolbeault operator

$$\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E),$$

where E is the underlying C^∞ bundle. This operator extends to give

$$\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E),$$

and has the following properties:

(i) $\bar\partial_E(f.s) = \bar\partial f.s + f.\bar\partial_E s$ for all $f \in C^\infty(X), s \in \Omega^0(E)$, where $\bar\partial$ is the usual Dolbeault operator on X.

(ii) $\bar\partial_E^2 = 0$.

Conversely, suppose we have a C^∞ complex vector bundle $E \to X$ equipped with an operator

$$\bar\partial_\alpha : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E),$$

satisfying (i) and the *integrability condition*:

1.3.1. $$\bar\partial_\alpha^2 = 0.$$

A well-known theorem of Koszul-Malgrange (see Chapter 2 of [21], for instance) shows that there then exists a holomorphic structure $\mathcal{E}$ on E whose Dolbeault operator satisfies

$$\bar\partial_{\mathcal{E}} = \bar\partial_\alpha.$$

Now suppose X is equipped with a Hermitian metric, whose associated $(1,1)$ form we denote by ω. The Hermitian structure induces the following decompositions of the self-dual and anti-self-dual two forms on X:

1.3.2. $$\Lambda_+^2 \cong \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbf{C}\{\omega\};$$

$$\Lambda_-^2 \cong \Lambda_0^{1,1},$$

where $\Lambda_0^{1,1}$ denotes the space of (1,1) forms orthogonal to ω. Let $E \to X$ be an SU(2) bundle, then a connection A on E induces a Dolbeault operator $\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)$ defined to be the composition

$$\Omega^0(E) \xrightarrow{\nabla_A} \Omega^1(E) \longrightarrow \Omega^{0,1}(E),$$

where the last map is orthogonal projection. There is a natural extension to

$$\bar\partial_A : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E),$$

with $\bar\partial_A^2 = F_A^{0,2}$, where F_A is the curvature of A and $F_A^{0,2}$ denotes its component in $\Omega^{0,2}(\mathrm{End}(E))$. The operator $\bar\partial_A$ satisfies the integrability condition (1.3.1) if and only if

1.3.3. $$F_A^{0,2} = 0.$$

In this case there is a holomorphic structure $\mathcal{E} = (E, \bar\partial_E)$ on E, such that $\bar\partial_A = \bar\partial_E$. Since the connection A preserves the trivialisation of $\Lambda^2 E$, we also have $\Lambda^2\mathcal{E} \cong \mathcal{O}_X$; i.e., $\mathcal{E}$ is an SL(2, C) bundle.

For a connection A on E, under the decomposition (1.3.2) the anti-self-duality equation,

$$F_A = - * F_A$$

splits into two equations:

1.3.4. $$F_A^{2,0} = F_A^{0,2} = 0;$$

1.3.5. $$\widehat{F_A} = 0,$$

where $\widehat{F_A}$ is the section of End(E) defined by

1.3.6. $$\widehat{F_A} = *(F_A \wedge \omega).$$

Equation (1.3.4) is just the integrability condition for the complex structure given by A, so any anti-self-dual connection on E defines a holomorphic bundle $\mathcal{E}$. Let $\mathcal{A}^{1,1}$ denote the set of SU(2) connections on E satisfying (1.3.4), i.e., those connections whose curvatures are of type $(1,1)$. The space $\mathcal{A}^{1,1}$ is then precisely the set of SU(2) connections which induce holomorphic structures on E. The action of $\mathcal{G}$ on $\mathcal{A}$ preserves $\mathcal{A}^{1,1}$ and extends to an action of the complexified group $\mathcal{G}^c$, defined by

1.3.7. $$\overline{\partial}_{g.A} = g \circ \overline{\partial}_A \circ g^{-1};$$

$$\partial_{g.A} = (g^*)^{-1} \circ \partial \circ (g^*),$$

for $g \in \mathcal{G}^c$, where g^* is the adjoint of g. The group $\mathcal{G}^c$ is the group of complex linear automorphisms of E which preserve the isomorphism $\Lambda^2 E \cong \mathbf{C}$. Two connections in $\mathcal{A}^{1,1}$ define the same holomorphic structure on E if and only if they are equivalent under the action of $\mathcal{G}^c$. Hence the space of holomorphic structures on E is parameterised by $\mathcal{A}^{1,1}/\mathcal{G}^c$. (This space is generally non-Hausdorff, but in our examples we shall consider only an open subspace $\mathcal{A}_s^{1,1} \subset \mathcal{A}^{1,1}$ preserved by $\mathcal{G}^c$ such that the quotient $\mathcal{A}_s^{1,1}/\mathcal{G}^c$ is a smooth manifold). The natural inclusion $\mathcal{B} \hookrightarrow \mathcal{A}^{1,1}$ then defines a map

1.3.8. $$\Phi : \mathcal{M}_k \to \mathcal{A}^{1,1}/\mathcal{G}^c.$$

Conversely, given a holomorphic SL$(2,\mathbf{C})$ bundle $\mathcal{E}$ on X with a Hermitian metric on its fibres, there is a unique SU(2) connection A on E such that $\overline{\partial}_E = \overline{\partial}_A$ [30, p.73]. The integrability condition (1.3.3) shows that $F_A^{2,0} = F_A^{0,2} = 0$. Hence, $A \in \mathcal{A}^{1,1}$ is anti-self-dual if and only if equation (1.3.5) is satisfied. We discuss equation (1.3.5) in the next section.

1.4. Stability and Hermitian-Einstein connections

If ω is a Kähler form on X, then there is a well-defined notion of the degree of a holomorphic vector bundle (or coherent sheaf) on X, and of stability.

The correspondence between stable holomorphic structures and Yang-Mills connections on bundles over Kähler manifolds has been studied in some detail. (See, for instance, [19],[21],[34] and [45].)

In the case of $S^1 \times S^3$, the Hermitian form ω is not closed, but does satisfy $\bar\partial\partial\omega = 0$. Buchdahl [18] has shown that this is enough to define a suitable notion of stability, as we now explain.

Suppose X is a closed Hermitian surface whose Hermitian form ω is $\bar\partial\partial$-closed. (In this case we call ω a *Gauduchon* metric). Given any coherent sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules on X, there is a well defined determinant line bundle $\det\mathcal{S}$ on X [34]. If we put a Hermitian metric h on $\det\mathcal{S}$, we obtain a unique connection on $\det\mathcal{S}$ whose curvature form F is given locally with respect to some trivialisation of $\det\mathcal{S}$ by $\bar\partial\partial\log h$ [30].

Definition 1.4.1. We define the *degree* of $\mathcal{S}$, $\deg\mathcal{S}$ by

$$\deg\mathcal{S} = \frac{\kappa}{2\pi i}\int_X F \wedge \omega,$$

where $\kappa \in \mathbf{R}_{>0}$ is a normalising constant (to be chosen appropriately).

A priori, this definition depends on the choice of Hermitian metric on $\det\mathcal{S}$, but for any two choices of metric, the resulting curvatures differ by a term of the form $i\bar\partial\partial\mu$ for some real function μ on X. Since $\bar\partial\partial\omega = 0$, an application of Stoke's theorem shows this term contributes nothing to the integral, so the degree of $\mathcal{S}$ is well-defined, and agrees with the usual definition when X is Kähler. Moreover, it behaves in the usual manner with respect to exact sequences of sheaves on X [18]. Unlike the Kähler case, however, the degree of a bundle depends on its holomorphic structure and not just on the topology of the underlying C^∞ bundle.

We now define the idea of *stability*.

Definition 1.4.2. A holomorphic bundle $\mathcal{E}$ on X is *stable* if whenever $\mathcal{S}$ is a coherent subsheaf of $\mathcal{E}$ with $0 < \operatorname{rank}\mathcal{S} < \operatorname{rank}\mathcal{E}$, we have

$$\frac{\deg\mathcal{S}}{\operatorname{rank}\mathcal{S}} < \frac{\deg\mathcal{E}}{\operatorname{rank}\mathcal{E}} \stackrel{\text{def}}{=} \mu(E).$$

Definition 1.4.3. Let $\mathcal{E} = (E, \bar\partial_E)$ be a holomorphic vector bundle over X with a Hermitian metric on its fibres. A connection A on E is called *Hermitian-Einstein* if it is compatible with the unitary structure and satisfies

$$\widehat{F_A} = i\mu.\mathrm{id}_E,$$

for some constant $\mu \in \mathbf{C}$.

Remark 1.4.4. The constant μ is uniquely determined; we have

$$\mu = \frac{2\pi}{\kappa.\mathrm{vol}(X)}\mu(\mathcal{E}).$$

In the case when $\mathcal{E}$ is an SL(2, C) bundle, $\mu(\mathcal{E}) = 0$ and a Hermitian-Einstein connection on $\mathcal{E}$ satisfies (1.3.3) and (1.3.5), i.e., it is anti-self-dual.

If E is an SU(2) bundle, equations (1.3.4) and (1.3.5) and Remark (1.4.4) show that the image of the natural map (1.3.8) is the space of orbits of Hermitian-Einstein connections on E under $\mathcal{G}^c$. The question of just how 'large' this subspace is, is answered by the following theorem of Buchdahl [18]:

Theorem 1.4.5 (Buchdahl). *If X is a closed Hermitian surface whose Hermitian form ω satisfies $\bar{\partial}\partial\omega = 0$, then a holomorphic bundle $\mathcal{E}$ with Hermitian metric admits an irreducible Hermitian-Einstein connection if and only if $\mathcal{E}$ is stable. The connection is then unique.*

Now let X denote $S^1 \times S^3$ equipped with the complex structure and metric described in Section (1.1). As we saw there, the metric on X is Hermitian and its associated (1,1) form ω satisfies $\bar{\partial}\partial\omega = 0$. The results of Sections (1.3) and above therefore apply. Suppose $E \to X$ is an SU(2) bundle with $c_2(E) = k > 0$. Since $H^2(X;\mathbf{Z}) = 0$, all connections on E are irreducible. Furthermore, the metric on X is conformally flat and has positive scalar curvature. Theorem 1.2.4 then shows that $\mathcal{M}_k$ is an $8k$-dimensional manifold. Denote by $\mathcal{A}^{1,1}_s$ the open subset of $\mathcal{A}^{1,1}$ consisting of stable holomorphic structures on E. As a result of the Buchdahl's theorem, Braam and Hurtubise proved the following:

Proposition 1.4.6 ([16], Prop.3.4.3). *For $k > 0$, the natural map $\mathcal{B} \hookrightarrow \mathcal{A}^{1,1}$ induces a diffeomorphism $\mathcal{M}_k \to \mathcal{A}^{1,1}_s/\mathcal{G}^c$.*

This map endows $\mathcal{M}_k$ with the structure of a $4k$-dimensional complex manifold. The Riemannian and complex structures of X also mean that we have two different groups acting on $\mathcal{M}_k$. As already described in Section 1.1, the group $S^1 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ acts on X by isometries and hence also on $\mathcal{M}_k$. In a similar way, the identity component $\mathrm{Aut}_0(X)$ of the complex automorphism group acts holomorphically on $\mathcal{A}^{1,1}_s/\mathcal{G}^c$, and hence by (1.4.6) on $\mathcal{M}_k$. This action will prove useful in the sequel.

In summary, Proposition 1.4.6 shows that study of anti-self-dual connections on E is equivalent to the study of stable holomorphic $\mathrm{SL}(2,\mathbf{C})$ structures on E. Hence our problem has been translated into one of complex analysis.

1.5. Some sheaf theory

Since we shall heavily use the language of coherent sheaves, we state below some of the basic results used. For further details and proofs, as well as all the necessary material on sheaf cohomology, see [11], [39], Chapter 3 of [32] and Chapter 5 of [34].

Recall that if $\mathcal{S}$ is a coherent sheaf of $\mathcal{O}_X$-modules over a complex manifold X then its singularity set $\sigma(\mathcal{S}) \subset X$ is defined by

$$\sigma(\mathcal{S}) = \{x \in X : \mathcal{S}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\},$$

and is an analytic subset of X of codimension at least 1. We say that $\mathcal{S}$ is *torsion free* if for each $x \in X$, the stalk $\mathcal{S}_x$ is a torsion-free $\mathcal{O}_{X,x}$-module. In this case $\sigma(\mathcal{S})$ is of codimension at least 2. In particular, any torsion free coherent sheaf on a Riemann surface is locally free.

A coherent sheaf is *reflexive* if $\mathcal{S} \cong \mathcal{S}^{**}$, where $\mathcal{S}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{S}, \mathcal{O}_X)$. Equivalently $\mathcal{S}$ is reflexive if and only if it is torsion free and *normal*, i.e., for every open set $U \subset X$ and every analytic subset $V \subset U$ of codimension at least 2, the restriction map

$$H^0(U;\mathcal{S}) \to H^0(U \setminus V;\mathcal{S})$$

is an isomorphism. Note that if $\mathcal{S}$ and $\mathcal{F}$ are coherent, with $\mathcal{F}$ reflexive, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{S},\mathcal{F})$ is also reflexive. In particular the dual $\mathcal{S}^*$ of $\mathcal{S}$ is reflexive.

If $\mathcal{S}$ is a coherent sheaf on X and there exists some $r \geq 1$ such that for each $x \in X$ there exists an open neighbourhood U of x and an exact sequence

$$0 \to \mathcal{S}_U \to \mathcal{O}_U^{m_1} \to \mathcal{O}_U^{m_2} \to \ldots \to \mathcal{O}_U^{m_r},$$

then we say $\mathcal{S}$ is locally an r^{th} syzygy sheaf. In this case the singularity set $\sigma(\mathcal{S})$ of $\mathcal{S}$ is of codimension at least $r+1$.

The following lemma taken from [39] (see also [19]) applies some of these results and is useful in the sequel.

Lemma 1.5.1. *Let $\mathcal{E}$ be a holomorphic* $\mathrm{SL}(2,\mathbf{C})$ *bundle over a closed complex surface X equipped with a Gauduchon metric. Then $\mathcal{E}$ is stable if and only if there is no line bundle $\mathcal{L}$ admitting a non-trivial map to $\mathcal{E}$ with* $\deg \mathcal{L} \geq 0$.

Proof of 1.5.1: We need only show that if $\mathcal{S}$ is a rank 1 coherent subsheaf of $\mathcal{E}$ with $\deg \mathcal{S} \geq 0$, then there is a line bundle $\mathcal{L}$ with $\deg \mathcal{L} \geq 0$, which admits a non-trivial map to $\mathcal{E}$. Since $\mathcal{S}$ is a subsheaf of $\mathcal{E}$ it is torsion free, and so is locally free outside a codimension 2 subset Y of X. Thus the line bundle $\mathcal{L} = \det\mathcal{S}$ is isomorphic to $\mathcal{S}$ on $X \setminus Y$. Since $\mathcal{E}$ is reflexive, the natural map $\mathcal{L} \longrightarrow \mathcal{E}$ defined on $X \setminus Y$ extends over Y, so $\mathcal{L}$ maps non-trivially to $\mathcal{E}$ and $\deg\mathcal{L} = \deg \mathcal{S} \geq 0$. $\boxed{1.5.1}$

In the next chapters we investigate stable bundles on X in detail.

Chapter 2

Principal elliptic surfaces

Braam and Hurtubise [16] studied stable holomorphic $\mathrm{SL}(2,\mathbf{C})$ bundles over Hopf surfaces by means of a 'graph' or 'divisor' invariant. Here we give an extension of their invariant to $\mathrm{SL}(2,\mathbf{C})$ bundles over a general non-trivial principal elliptic fibration and study its properties. We shall need it in this generality in later chapters, when we come to consider bundles on principal elliptic surfaces over curves other than $\mathbf{P}^1$.

2.1. Line bundles on elliptic curves

All our elliptic curves shall be of the form $\mathbf{C}^*/\mathbf{Z}$ where the $\mathbf{Z}$ action on $\mathbf{C}^*$ is generated by the dilation $z \mapsto \lambda z$, with $\lambda \in \mathbf{R}$ and $\lambda > 1$. More commonly, we shall fix λ and denote this curve simply by T. We give T the structure of an Abelian variety by choosing the point determined by $1 \in \mathbf{C}^*$ as the identity element. Since $H^1(T;\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$, $H^2(T;\mathbf{Z}) \cong \mathbf{Z}$ and $H^1(T;\mathcal{O}_T) \cong \mathbf{C}$,we have:

$$\mathrm{Pic}^0(T) \cong H^1(T;\mathcal{O}_T)/H^1(T;\mathbf{Z}) \cong \mathbf{C}^*/\mathbf{Z} \cong T;$$

$$\mathrm{Pic}(T) \cong \mathrm{Pic}^0(T) \times \mathbf{Z}.$$

The choice of identity for T gives the isomorphism $\mathrm{Pic}^0(T) \cong T$ [32]. Explicitly we can see the isomorphism as follows: a specific representative $\mathcal{V}_\mu$ for the isomorphism class of line bundles represented by $\mu \in \mathbf{C}^*$ can be constructed by taking the quotient of $\mathbf{C}^* \times \mathbf{C}$ by the action of $\mathbf{Z}$ generated by

$$(z,\xi) \mapsto (\lambda z, \mu\xi).$$

If $\mathcal{V}_\mu \cong \mathcal{V}_{\mu'}$ for $\mu, \mu' \in \mathbf{C}^*$ then $\mu = \lambda^n \mu'$ for some $n \in \mathbf{Z}$, giving the identification of $\mathrm{Pic}^0(T)$ with $\mathbf{C}^*/\mathbf{Z} = T$. We shall usually make this identification for convenience.

The map $z \mapsto z^{-1}$ on $\mathbf{C}^*$ descends to give a holomorphic involution ι on T, the usual "hyperelliptic involution". Geometrically ι corresponds to rotation of an angle π about the axis of the torus passing through the four half-periods.

The quotient of T by the $\mathbf{Z}_2$ action is isomorphic to $\mathbf{P}^1$ and we have a double cover

$$q : T \to \mathbf{P}^1,$$

ramified at the half-periods $\pm 1, \pm\sqrt{\lambda}$ in T.

2.2. Principal elliptic bundles

Let $\pi : Y \to B$ be a principal elliptic fibration, where B and Y are closed complex manifolds of dimensions $n, n+1$ respectively, with fibre the elliptic curve T. Then, as in [12], Y is given by an element of the cohomology group $H^1(B; \mathcal{T})$, where $\mathcal{T}$ is the sheaf of germs of holomorphic maps from B to the group of translations of T. (Note that the latter group is naturally identified with T itself). Thus there is an open cover $\{U_\alpha\}$ of B such that Y is isomorphic to the quotient of $\sqcup U_\alpha \times T$ by the relation generated by $(z, t) \sim (z, g_{\alpha\beta}(z)t)$ for some Čech cocycle $\{g_{\alpha\beta}\}$ representing a class in $H^1(\{U_\alpha\}; \mathcal{T})$.

Alternatively, there is a line bundle $\mathcal{L}$ over B such that the quotient of the $\mathbf{C}^*$ bundle $\mathcal{L}^*$, obtained by removing the zero section of $\mathcal{L}$, by the $\mathbf{Z}$ action generated by scalar multiplication by $\lambda \in \mathbf{C}^*$ in each fibre is isomorphic to Y. (The transition functions of $\mathcal{L}$ are lifts to $\mathbf{C}^*$ of the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to T$). For further details of principal elliptic fibrations, we refer to [12].

In the case when Y is a non-trivial principal elliptic surface, its holomorphic and topological invariants are summarised in the following lemma.

Lemma 2.2.1. *A non-trivial principal elliptic surface $\pi : Y \to B$ over a closed compact Riemann surface B of genus g has the following invariants:*

$$h^{0,1}(Y) = g + 1; \qquad h^{0,2}(Y) = g;$$
$$H^1(Y; \mathbf{Z}) \cong \mathbf{Z}^{2g+1}; \qquad H^2(Y; \mathbf{Z}) \cong \mathbf{Z}^{4g} \oplus \mathbf{Z}_m,$$

where m is the degree of the holomorphic line bundle over B associated to Y. The intersection form is negative definite on restriction to the image of $\mathrm{Pic}(Y)$ *in $H^2(Y; \mathbf{Z})/H^2(Y; \mathbf{Z})_{\mathrm{tors}}$.*

Proof of 2.2.1: Topologically Y is $C \times S^1$, where C is a circle bundle over B with Euler class of degree m. The associated Gysin sequence [12, p.145] gives exact sequences

$$0 \to H^1(B; \mathbf{Z}) \to H^1(C; \mathbf{Z}) \to 0;$$

$$0 \to H^0(B; \mathbf{Z}) \xrightarrow{m} H^2(B; \mathbf{Z}) \to H^2(C; \mathbf{Z}) \to H^1(B; \mathbf{Z}) \to 0,$$

and this together with the Künneth theorem gives the the topological invariants. The canonical bundle formula of Kodaira [12, p.161] gives $K_Y \cong \pi^* K_B$. The integer $h^{0,2}(Y)$ is then determined by Serre duality and the value of $h^{0,1}(Y)$ follows by Riemann-Roch. The statement about the intersection form follows from the signature theorem [12, p.120] since $b_1(Y)$ is odd. $\boxed{2.2.1}$

Let $\mathrm{Pic}^0(Y, B)$ denote the subgroup of $\mathrm{Pic}(Y)$ consisting of line bundles which restrict to bundles of degree 0 on all fibres of π. The following construction was given in [16]. Using the first description of Y, given $\mathcal{L} \in \mathrm{Pic}^0(Y, B)$ we can restrict it to a local trivialisation $U_\alpha \times T$. Restricting to the fibre $\pi^{-1}(x)$ above $x \in U_\alpha$ defines a map $\phi_\alpha : U_\alpha \to \mathrm{Pic}^0(T)$. If $x \in U_\alpha \cap U_\beta$ then $\phi_\alpha(x)$ and $\phi_\beta(x) \in \mathrm{Pic}^0(T)$ differ by the action of translation by $g_{\alpha\beta}(x)$ on $\mathrm{Pic}^0(T)$. But this action is trivial [30, p.312], and so ϕ_α and ϕ_β agree on $U_\alpha \cap U_\beta$. Thus we obtain a well-defined homomorphism

2.2.2. $$\phi : \mathrm{Pic}^0(Y, B) \to \mathrm{Map}(B, \mathrm{Pic}^0(T)) = \mathrm{Map}(B, T),$$

where $\mathrm{Map}(B, T)$ denotes the group of holomorphic maps from B to the Abelian variety T.

Definition 2.2.3. The graph of the map $\phi(\mathcal{L}) : B \to T$ defined above is called the *graph* of the line bundle $\mathcal{L}$.

It is natural to ask if given a holomorphic map $g : B \to T$ whether or not there is a line bundle $\mathcal{L} \in \mathrm{Pic}^0(Y, B)$ whose graph is that of g. We describe briefly the obstruction to the existence of $\mathcal{L}$ and refer to [31] for further details. Pushing down the exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$$

gives an exact sequence

2.2.4. $$0 \to R^1\pi_*\mathbf{Z} \to R^1\pi_*\mathcal{O}_Y \to R^1\pi_*\mathcal{O}_Y^* \to \mathbf{Z}.$$

Since all fibres of π are non-singular, the locally constant sheaf $R^1\pi_*\mathbf{Z}$ is closed in $R^1\pi_*\mathcal{O}_Y$ and the quotient $R^1\pi_*\mathcal{O}_Y/R^1\pi_*\mathbf{Z} = \mathcal{J}ac(Y, B)$ is precisely the sheaf of sections of the Jacobian variety $\mathrm{Jac}(Y, B)$ associated to the elliptic fibration Y [12, p.153]. In our case, $\mathrm{Jac}(Y, B) \cong B \times T$ and the space $\mathrm{Map}(B, T)$ is just $H^0(\mathcal{J}ac(Y, B))$. Following Grothendieck, denote the group $H^0(R^1\pi_*\mathcal{O}_Y^*)$ by $\mathrm{Pic}(Y/B)$. (The sheaf $R^1\pi_*\mathcal{O}_Y^*$ is the sheaf of sections of the relative Picard variety [31].) We then have an injection

2.2.5. $$H^0(R^1\pi_*\mathcal{O}_Y/R^1\pi_*\mathbf{Z}) = \mathrm{Map}(B, T) \hookrightarrow \mathrm{Pic}(Y/B)$$

induced by the exact sequence

$$0 \to \mathcal{J}ac(Y, B) \to R^1\pi_*\mathcal{O}_Y^* \to \mathbf{Z}$$

coming from (2.2.4). The lower term sequence of the Leray spectral sequence for $\mathcal{O}_Y^*$ with $E_2^{p,q} = H^p(B; R^q\pi_*\mathcal{O}_Y^*)$ gives an exact sequence

2.2.6. $$0 \to H^1(B; \mathcal{O}_B^*) \to H^1(Y; \mathcal{O}_Y^*) \to H^0(B; R^1\pi_*\mathcal{O}_Y^*) \xrightarrow{\delta} H^2(B; \mathcal{O}_B^*),$$

(using $\pi_*\mathcal{O}_Y^* = \mathcal{O}_B^*$); i.e.,

$$0 \to \mathrm{Pic}(B) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(Y/B) \xrightarrow{\delta} \mathrm{Br}'(B),$$

where $\mathrm{Br}'(B) = H^2(B; \mathcal{O}_B^*)$ is the *cohomological Brauer group* [31]. Hence we have:

Lemma 2.2.7. *An element η of* Map(B,T) *lifts to give a globally defined line-bundle on Y if and only if η maps to* 0 *in* $\mathrm{Br}'(B)$ *under the composition*

$$\mathrm{Map}(B,T) \hookrightarrow \mathrm{Pic}(Y/B) \xrightarrow{\delta} \mathrm{Br}'(B).$$

As a corollary we have:

Corollary 2.2.8. *If $H^2(B,\mathcal{O}_B^*) = 0$ then the map $\phi : \mathrm{Pic}^0(Y,B) \to$* Map$(B,T)$ *is surjective.*

In this case there is a rather explicit construction of such a line bundle. Choose a Leray cover $\{U_\alpha\}$ for $\mathcal{O}_B^*$ such that $Y|U_\alpha \cong U_\alpha \times T$ for each α. Let $g : B \to T$ be given and let $g_\alpha : U_\alpha \to T$ be the restriction of g to U_α. Define a line bundle $\mathcal{L}_\alpha$ on $Y|U_\alpha$ to be the pull-back of a fixed Poincaré bundle $\mathcal{P} \to \mathrm{Pic}^0(T) \times T$ by the map

$$\Phi_\alpha : U_\alpha \times T \to \mathrm{Pic}^0(T) \times T,$$
$$(x,t) \to (g_\alpha(x),t).$$

On $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the line bundle $\pi_*(\mathcal{L}_\alpha^{-1}\mathcal{L}_\beta)$ is trivial so there is an isomorphism

$$\phi_{\alpha\beta} : \mathcal{L}_\alpha|\pi^{-1}(U_{\alpha\beta}) \to \mathcal{L}_\beta|\pi^{-1}(U_{\alpha\beta}).$$

We wish to define a line bundle $\mathcal{L}$ on Y by gluing the $\mathcal{L}_\alpha$ together using the local isomorphisms $\phi_{\alpha\beta}$. We may do so provided the condition:

$$\rho_{\alpha\beta\gamma} \stackrel{\mathrm{def}}{=} \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1,$$

is satisfied. The Čech 2-cocycle $\rho_{\alpha\beta\gamma}$ defines an element in $H^2(B;\mathcal{O}_B^*)$. If $H^2(B;\mathcal{O}_B^*) = 0$, there is a 1-cochain $(\theta_{\alpha\beta}) \in C^1(\{U_\alpha\};\mathcal{O}_B^*)$ such that

$$\rho_{\alpha\beta\gamma} = \theta_{\alpha\beta}\theta_{\beta\gamma}\theta_{\gamma\alpha}.$$

Defining $\phi'_{\alpha\beta} = (\theta_{\alpha\beta} \circ \pi)^{-1}\phi_{\alpha\beta}$ gives isomorphisms

$$\phi'_{\alpha\beta} : \mathcal{L}_\alpha|\pi^{-1}(U_{\alpha\beta}) \to \mathcal{L}_\beta|\pi^{-1}(U_{\alpha\beta}),$$

with

$$\phi'_{\alpha\beta}\phi'_{\beta\gamma}\phi'_{\gamma\alpha} = 1.$$

Hence we may glue the $\{\mathcal{L}_\alpha\}$ together to obtain a bundle $\mathcal{L} \in \mathrm{Pic}^0(Y,B)$ with the required graph.

As a further corollary of (2.2.7) we obtain:

Corollary 2.2.9. *If $\pi : Y \to B$ is a principal elliptic surface then the map $\phi : \mathrm{Pic}^0(Y,B) \to \mathrm{Map}(B,T)$ is surjective and we have an exact sequence:*

2.2.10. $$1 \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}^0(Y,B) \longrightarrow \mathrm{Map}(B,T) \longrightarrow 1.$$

Proof of 2.2.9: The first statement follows since $H^2(B;\mathcal{O}_B^*) = 0$ and corollary (2.2.8). The last part follows for if $\mathcal{L} \in \mathrm{Pic}^0(Y,B)$ is such that $\phi(\mathcal{L}) = 1$, then $\mathcal{L}$ is trivial on fibres of π and so must be the pull-back of some line bundle on B. $\boxed{2.2.9}$

Definition 2.2.11. Let $\mathrm{Pic}^s(Y,B)$ denote the subgroup of $\mathrm{Pic}^0(Y,B)$ which maps to the subgroup T of $\mathrm{Map}(B,T)$ consisting of constant maps.

We therefore have an exact sequence

2.2.12. $$1 \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}^s(Y,B) \longrightarrow T \longrightarrow 1.$$

Definition 2.2.13. Let $\mathrm{Pic}^\tau(Y)$ denote the subgroup of $\mathrm{Pic}(Y)$ consisting of elements $\mathcal{V}$ such that $c_1(\mathcal{V}) \in H^2(Y;\mathbf{Z})_{\mathrm{tors}}$.

There is an obvious exact sequence,

2.2.14. $$1 \to \mathrm{Pic}^0(Y) \to \mathrm{Pic}^\tau(Y) \to H^2(Y;\mathbf{Z})_{\mathrm{tors}} \to 0.$$

Remark 2.2.15. Elements of $\mathrm{Pic}^s(Y,B)$ can be constructed easily using our second description of Y as the quotient space $\mathcal{L}^*/\mathbf{Z}$ where $\mathcal{L}$ is a line bundle over B. Given $\mu \in \mathbf{C}^*$, one defines a line bundle $\mathcal{L}_\mu$ on Y to be the quotient of $\mathcal{L}^* \times \mathbf{C}$ by the $\mathbf{Z}$-action generated by $(l,\xi) \mapsto (\lambda l, \mu\xi)$. The graph of $\mathcal{L}_\mu$ is then the constant map $(\mu) \in T$. Since $\mathcal{L}_\mu$ has constant transition functions it is flat as a C^∞-bundle and hence also an element of $\mathrm{Pic}^\tau(Y)$.

The following result is a standard corollary of the base change theorem (see [11], [32] or [38]), whose relevance will soon become apparent.

Lemma 2.2.16. *Suppose $\pi : Y \to B$ is a proper holomorphic fibration of complex spaces and let $\mathcal{F}$ be a coherent sheaf over Y which is flat with respect to π. If for some $q \in \mathbf{Z}$ the direct image sheaf $R^q\pi_*\mathcal{F} = 0$, then the natural evaluation map*
$$(R^{q-1}\pi_*\mathcal{F})_x \to H^{q-1}(\pi^{-1}(x);\mathcal{F})$$
is a surjection for all $x \in B$.

We apply this result in the case where $\pi : Y \to B$ is a *non-trivial principal elliptic surface* to give an alternative construction of the graph of a line bundle, which coincides with that of [16]. Suppose we are given $\mathcal{L} \in \mathrm{Pic}^0(Y,B)$ and let $\mathcal{L}_\mu$ be the bundle constructed in Remark 2.2.15 for some $\mu \in \mathbf{C}^*$. We then have either
$$h^0(\pi^{-1}(x);\mathcal{L}_\mu^{-1}\mathcal{L}) = 1, \text{ for all } x \in B;$$
or

2.2.17. $$h^0(\pi^{-1}(x);\mathcal{L}_\mu^{-1}\mathcal{L}) = 0, \text{ generically}.$$

In the first case, the sheaf $\pi_*(\mathcal{L}_\mu^{-1}\mathcal{L})$ is torsion free of rank 1 and hence corresponds to a line bundle $\mathcal{U}$ on B. Since the fibres of π are all irreducible we obtain $\mathcal{L} \cong \mathcal{L}_\mu \otimes \pi^*\mathcal{U}$.

In the second case, the restrictions of $\mathcal{L}$ and $\mathcal{L}_\mu$ to a generic fibre are non-isomorphic. However they are isomorphic on fibres $\pi^{-1}(x)$ for which $h^1(\pi^{-1}(x); \mathcal{L}_\mu^{-1}\mathcal{L}) = 1$. Since the fibres of π are 1-dimensional, the sheaf $R^2\pi_*(\mathcal{L}_\mu^{-1}\mathcal{L})$ vanishes and so by Lemma 2.2.16 the sheaf $R^1\pi_*(\mathcal{L}_\mu^{-1}\mathcal{L})$ is a skyscraper sheaf supported precisely at those points $x \in B$ for which $h^1(\pi^{-1}(x); \mathcal{L}_\mu^{-1}\mathcal{L}) = 1$.

Lemma 2.2.18. *With the above notation $R^1\pi_*(\mathcal{L}_\mu^{-1}\mathcal{L})$ is a skyscraper sheaf supported on a divisor of degree $-\frac{1}{2}c_1(\mathcal{L})^2$ on B.*

Proof of 2.2.18: This follows from the Grothendieck-Riemann-Roch theorem [26, 32] using the fact that $\mathcal{L}_\mu$ is flat. $\boxed{2.2.18}$

We can now identify the subgroup $\mathrm{Pic}^s(Y, B)$ in topological terms.

Lemma 2.2.19. *The subgroups $\mathrm{Pic}^s(Y, B)$ and $\mathrm{Pic}^\tau(Y)$ coincide when Y is a non-trivial principal elliptic surface.*

Proof of 2.2.19: First we show that $\mathrm{Pic}^\tau(Y) \subset \mathrm{Pic}^s(Y, B)$. If $\mathcal{L} \in \mathrm{Pic}^\tau(Y)$ then its restriction to each fibre of π is of degree 0. Lemma 2.2.18 and the fact that $c_1(\mathcal{L})^2 = 0$ show that the case 2.2.17 cannot occur. Hence by the discussion above, $\mathcal{L}$ is isomorphic to $\mathcal{L}_\mu \otimes \pi^*\mathcal{U}$ for some $\mu \in \mathbf{C}^*$ and line bundle $\mathcal{U}$ on B. The graph of $\mathcal{L}$ is then the constant $(\mu) \in T$.

Conversely, if $\mathcal{V} \in \mathrm{Pic}^s(Y, B)$ has graph (μ), then $\mathcal{V}$ is isomorphic to $\mathcal{L}_\mu \otimes \pi^*\mathcal{U}$ for some line bundle $\mathcal{U}$ on B. The flatness of $\mathcal{L}_\mu$ gives $c_1(\mathcal{V})^2 = 0$. The negative-definiteness of the intersection form on the image of $\mathrm{Pic}(Y)$ in $H^2(Y; \mathbf{Z})/H^2(Y; \mathbf{Z})_{\mathrm{tors}}$ then gives the result. $\boxed{2.2.19}$

Lemma 2.2.20. *Under the hypotheses of Lemmas (2.2.1) and (2.2.19) there is an isomorphism*

$$\mathrm{Pic}^\tau(Y) \cong \pi^*\mathrm{Pic}^0(B) \times \mathbf{C}^* \times \mathbf{Z}_m.$$

Proof of 2.2.20: We have a commuting diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^1(B;\mathbf{Z}) \arrow[r] \arrow[d, "\pi^*"'] & H^1(B;\mathcal{O}_B) \arrow[r] \arrow[d, "\pi^*"'] & H^1(B;\mathcal{O}_B^*) \arrow[r] \arrow[d, "\pi^*"'] & {} \\
0 \arrow[r] & H^1(Y;\mathbf{Z}) \arrow[r] \arrow[d] & H^1(Y;\mathcal{O}_Y) \arrow[r] \arrow[d] & H^1(Y;\mathcal{O}_Y^*) \arrow[r] & {} \\
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d] & \mathbf{C} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the first two columns arise from the Gysin sequence with coefficients in $\mathbf{Z}$ and $\mathbf{C}$ respectively. The diagram gives rise to an exact sequence

$$0 \to H^1(B;\mathcal{O}_B)/H^1(B;\mathbf{Z}) \xrightarrow{\pi^*} H^1(Y;\mathcal{O}_Y)/H^1(Y;\mathbf{Z}) \to \mathbf{C}/\mathbf{Z} \to 0,$$

i.e.,

$$0 \to \mathrm{Pic}^0(B) \xrightarrow{\pi^*} \mathrm{Pic}^0(Y) \to \mathbf{C}^* \to 0.$$

Hence each element of $\mathrm{Pic}^0(Y)$ can be written in the form $\mathcal{L}_\mu \otimes \pi^*\mathcal{U}$ for some $\mu \in \mathbf{C}^*$ and $\mathcal{U} \in \mathrm{Pic}^0(B)$. The exact sequence (2.2.14) leads to the result. $\boxed{2.2.20}$

A choice of Poincaré line bundle for B and holomorphic line bundle on Y whose first Chern class generates $\mathbf{Z}_m$ allows us to construct a Poincaré line bundle $\mathcal{V} \to Y \times \mathrm{Pic}^s(Y,B)$. Let $\tilde{\pi} : Y \times \mathrm{Pic}^s(Y,B) \to B \times \mathrm{Pic}^s(Y,B)$ and $p_1 : Y \times \mathrm{Pic}^s(Y,B) \to Y$ be the obvious projections. We form the direct image sheaf $R^1\tilde{\pi}_*(\mathcal{V}^{-1}p_1^*\mathcal{L})$. This is supported on a divisor $\tilde{G}$ of $B \times \mathrm{Pic}^s(Y,B)$ with the following properties:

(i) A point $(x,\mathcal{U})$ lies on $\tilde{G}$ if and only if $\mathcal{L}|_{\pi^{-1}(x)} \cong \mathcal{U}|_{\pi^{-1}(x)}$.

(ii) The divisor $\tilde{G}$ is invariant under the action induced by tensoring $\mathrm{Pic}^s(Y,B)$ by elements of $\mathrm{Pic}(B)$.

Properties (i) and (ii) show that the divisor $\tilde{G}$ descends to a divisor G on $B \times \mathrm{Pic}^s(Y,B)/\mathrm{Pic}(B) \cong B \times T$. Since the divisor $\{b\} \times T$ intersects G in exactly one point for all $b \in B$, and generically $B \times \{t\}$ intersects G in $d = -\frac{1}{2}c_1(\mathcal{L})^2$ points by Grothendieck-Riemann-Roch (as above), the divisor G must be the graph of a degree d map from B to T. This definition clearly coincides with the earlier definition of the graph. As a corollary we have:

Corollary 2.2.21. *Suppose $\mathcal{L} \in \mathrm{Pic}^0(Y,B)$. Let $B_t = B \times \{t\}$ for $t \in T$. If the graph G of $\mathcal{L}$ is non-constant then $G.B_t = -\frac{1}{2}c_1(\mathcal{L})^2$, for all $t \in T$.*

2.3. SL(2, ℂ) bundles on principal elliptic surfaces

We recall the following theorem of Atiyah classifying SL(2, C) bundles on an elliptic curve [6, 32].

Theorem 2.3.1. *Suppose $\mathcal{E}$ is a holomorphic* SL(2, C) *bundle over an elliptic curve T, then either $\mathcal{E}$ is decomposable or $\mathcal{E}$ is a non-trivial extension*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0, \text{ with } \mathcal{L}^2 \cong \mathcal{O}_T.$$

Hence if $\mathcal{E}$ an SL(2, C) bundle over T it falls into one of the following three classes:

1 $\mathcal{E} \cong \mathcal{L}_0 \oplus \mathcal{L}_0^{-1}$, with $\mathcal{L}_0 \in \mathrm{Pic}^0(T)$;

2 A non-trivial extension:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L} \to 0,$$

with $\mathcal{L} \in \mathrm{Pic}^0(T)$ and $\mathcal{L}^2 \cong \mathcal{O}_T$;

3 $\mathcal{E} \cong \mathcal{L}_k \oplus \mathcal{L}_k^{-1}$ with $\mathcal{L}_k \in \mathrm{Pic}^k(T)$, $k > 0$.

Remark 2.3.2. Note that $\mathcal{E}$ is of type (3) if and only if for all $\mathcal{L}_0 \in \mathrm{Pic}^0(T)$, $H^0(T; \mathcal{L}_0^{-1}\mathcal{E}) \neq 0$.

We wish to examine certain SL(2, C) bundles on principal elliptic surfaces $\pi : Y \to B$, and in particular over Hopf surfaces. The classification of Atiyah enables us to analyse the bundle 'fibrewise', and we describe an extension of the 'graph' invariant of [16]. Note, however, that our notation is different.

Remark 2.3.3. Unless otherwise stated, in the sequel $\pi : Y \to B$ will denote a non-trivial principal elliptic surface over a smooth closed complex curve B.

Definition 2.3.4. Let $SL_0(Y, B)$ denote the set of rank 2 bundles $\mathcal{E}$ on Y which satisfy the following conditions:

(i) $\det \mathcal{E}|_{\pi^{-1}(x)}$ is trivial for all $x \in B$;

(ii) $\mathcal{E}|_{\pi^{-1}(x)}$ is of type (1) or (2) for generic $x \in B$.

Remark 2.3.5. Condition **(i)** is equivalent to the condition

(i)′ $\det \mathcal{E} \cong \pi^*\mathcal{U}$ for some line bundle $\mathcal{U}$ on B.

Definition 2.3.6. Let $SL_0^k(Y)$ denote the set of SL(2, C) bundles $\mathcal{E}$ on Y with $c_2(\mathcal{E}) = k$ which satisfy condition **(ii)** above.

Suppose $\mathcal{E} \in SL_0(Y, B)$ and that $c_2(\mathcal{E}) = k > 0$. Fix an element $\mathcal{L} \in \mathrm{Pic}^*(Y, B)$ such that for generic $x \in B$, $h^0(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) = 0$. Thus the direct image sheaf $\pi_*(\mathcal{L}^{-1}\mathcal{E}) = 0$. However, by (2.2.16), the first direct image sheaf $R^1\pi_*(\mathcal{L}^{-1}\mathcal{E})$ is a skyscraper sheaf supported at those points $x \in B$ for which $h^1(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) > 0$, i.e. for which $\mathcal{E}|_{\pi^{-1}(x)}$ is of type (1) or (2) with $\mathcal{L}|_{\pi^{-1}(x)}$ as a sub-bundle, or $\mathcal{E}|_{\pi^{-1}(x)}$ is of type (3). The sheaf $R^1\pi_*(\mathcal{L}^{-1}\mathcal{E})$ is coherent and since B is a curve, it always has local resolutions of length one. Thus, any point $x \in B$ has an open neighbourhood U with a local resolution

$$0 \to \mathcal{O}_U^m \xrightarrow{A} \mathcal{O}_U^m \to (R^1\pi_*\mathcal{L}^{-1}\mathcal{E})(U) \to 0,$$

where A is an $m \times m$ matrix of functions holomorphic on U. Locally the first Chern class of $R^1\pi_*(\mathcal{L}^{-1}\mathcal{E})$ is given by the zero-divisor of $\det A$ [16], [34]. Hence the divisor on which it is supported may be identified with $c_1(R^1\pi_*(\mathcal{L}^{-1}\mathcal{E}))$. By the Grothendieck-Riemann-Roch theorem, this is a divisor of degree k on B.

If we now take a Poincaré bundle $\mathcal{V} \to Y \times \mathrm{Pic}^*(Y, B)$ and consider $R^1\tilde{\pi}_*(\mathcal{V}\tilde{\mathcal{E}})$ where $\tilde{\pi} : Y \times \mathrm{Pic}^*(Y, B) \to B \times \mathrm{Pic}^*(Y, B)$ is the projection induced by π, and $\tilde{\mathcal{E}}$ is the pull-back of $\mathcal{E}$ to $Y \times \mathrm{Pic}^*(Y, B)$ induced by projection onto the first factor, then $R^1\tilde{\pi}_*(\mathcal{V}\tilde{\mathcal{E}})$ is a torsion sheaf on $Y \times \mathrm{Pic}^*(Y, B)$

supported on a divisor D. The points in the support of D are those points $(x, \mathcal{L}) \in B \times \mathrm{Pic}^s(Y, B)$ for which

$$h^0(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) > 0. \tag{2.3.7}$$

Hence D has the following properties:

(i) D is invariant under the action of $\mathrm{Pic}^s(B)$ on $B \times \mathrm{Pic}^s(Y, B)$ induced by tensor product with the second factor.

(ii) Since (2.3.7) remains true with $\mathcal{L}^{-1}$ replacing $\mathcal{L}$, D is invariant under the action of $\mathbf{Z}_2$ on $\mathrm{Pic}^s(Y, B)$.

Property (i) and the exact sequence (2.2.12) show that D descends to a divisor on

$$B \times \mathrm{Pic}^s(Y, B)/\mathrm{Pic}(B) = B \times T,$$

and property (ii) shows this divisor further descends to a divisor $D(\mathcal{E})$ on $B \times T/\mathbf{Z}_2 \cong B \times \mathbf{P}^1$.

Definition 2.3.8. The divisor $D(\mathcal{E})$ constructed above is called the *divisor* of $\mathcal{E}$.

Generically, by (2.3.7) the "horizontal fibres" $B \times \{l\}$ for $l \in \mathbf{P}^1$ contain k points of $D(\mathcal{E})$ counted with multiplicity. The "vertical fibres" $\{x\} \times \mathbf{P}^1$, for $x \in B$, generically contain just one point of $D(\mathcal{E})$. Otherwise the entire "vertical" fibre $\{x\} \times \mathbf{P}^1$ is included in the divisor (with some multiplicity). In this case $\mathcal{E}|_{\pi^{-1}(x)}$ is of type (3).

Consequently the divisor reduces into distinct components:

$$D(\mathcal{E}) = G_\alpha + \sum_{j=1}^{s} n_j F_j, \tag{2.3.9}$$

as illustrated in Figure 2.1. Here G_α is the graph of a holomorphic map $\alpha : B \to \mathbf{P}^1$ of degree r. The $F_j = \{x_j\} \times B$ are the vertical fibres over a set $\{x_1, \ldots, x_s\}$ of *distinct* points in B for which $\mathcal{E}|_{\pi^{-1}(x_j)}$ is of type (3), and $n_j \in \mathbf{Z}$ are their multiplicities. These quantities satisfy the relation:

$$r + \textstyle\sum_{j=1}^{s} n_j = k. \tag{2.3.10}$$

Definition 2.3.11. The divisor G_α on $B \times \mathbf{P}^1$ given by (2.3.9) is called the *graph* of $\mathcal{E}$.

2.4. Destabilising line bundles and exact sequences

Suppose now that $\mathcal{E} \in SL_0(Y, B)$ and $D(\mathcal{E})$ is of the form (2.3.9). The divisor G_α is the image under the quotient map $B \times T \to B \times \mathbf{P}^1$ of an ι invariant divisor $\tilde{G}$ on $B \times T$. Suppose

$$\tilde{G} = G_\rho + \iota^* G_\rho, \tag{2.4.1}$$

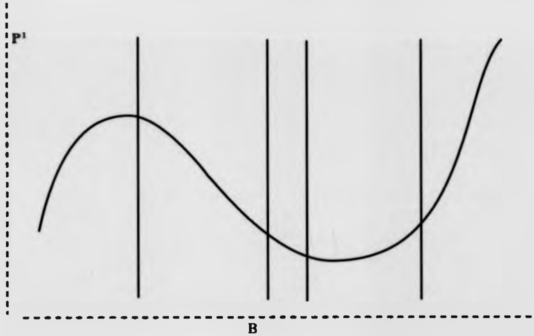

Figure 2.1. Divisor of an SL(2, C) bundle. Vertical components correspond to elliptic fibres where the bundle admits a sub-bundle of degree greater than 0

where G_ρ is the graph of a holomorphic map $\rho : B \to T$. By Corollary 2.2.8, we know there are line bundles $\mathcal{L}_0, \mathcal{L}_0' \in \mathrm{Pic}^0(Y, B)$ with graphs $G_\rho, \iota^* G_\rho$ respectively. Hence, on each fibre of π the bundles $\mathcal{L}_0, \mathcal{L}_0'$ restrict to sub-bundles of $\mathcal{E}$. One might then expect that any line bundle which maps to $\mathcal{E}$ is closely related to $\mathcal{L}_0$ or $\mathcal{L}_0'$. The following results are the analogues of Proposition 3.3.4 of [16].

Lemma 2.4.2. *Suppose $\mathcal{E} \in SL_0(Y, B)$ does not restrict to a bundle of type (1) with $\mathcal{L}_0^2 \cong \mathcal{O}$ on a generic fibre of π. If the divisor G_a can be written in the form (2.4.1) for some map $\rho : B \to T$, then there exist two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}^0(Y, B)$ such that any line bundle admitting a non-trivial map to $\mathcal{E}$ is isomorphic to one in the set*

$$\{\mathcal{L}_1 \otimes \pi^* \mathcal{O}_B(-D), \mathcal{L}_2 \otimes \pi^* \mathcal{O}_B(-D) : D \text{ an effective divisor on } B\}.$$

The line bundles satisfy $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \pi_*(\mathcal{L}_2^{-1}\mathcal{E}) \cong \mathcal{O}_B$. If $\mathcal{E}$ is of type (2) on a generic fibre, then $\mathcal{L}_1 \cong \mathcal{L}_2$.

Proof of 2.4.2: By Corollary 2.2.8 there exists a line bundle $\mathcal{L}_0 \in \mathrm{Pic}^0(Y, B)$ such that $\mathcal{L}_0$ has graph ρ. The hypotheses imply that for generic $x \in B$ we have $h^0(\pi^{-1}(x); \mathcal{L}_0^{-1}\mathcal{E}) = 1$. Hence $\pi_*(\mathcal{L}_0^{-1}\mathcal{E})$ is a rank 1 torsion-free sheaf on B and so isomorphic to some line bundle $\mathcal{W}$ say. Let $\mathcal{L}_1 = \mathcal{L}_0 \otimes \mathcal{W}$, then by the projection formula [32, p.124], $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \mathcal{O}_B$. Applying a similar argument to the map $\iota \circ \rho$ yields $\mathcal{L}_2$.

If $\mathcal{L}$ is any line bundle admitting a non-trivial map to $\mathcal{E}$, then without loss of generality we may assume $\mathcal{L}$ has the same graph as $\mathcal{L}_1$. Then $\pi_*(\mathcal{L}^{-1}\mathcal{L}_1)$ is a line bundle $\mathcal{U}$ on B, giving $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{U}^{-1}$. Now $h^0(B; \pi_*(\mathcal{L}^{-1}\mathcal{E})) = h^0(Y; \mathcal{L}^{-1}\mathcal{E}) \neq 0$, but by the projection formula $\pi_*(\mathcal{L}^{-1}\mathcal{E}) = \mathcal{U}$. Thus $\mathcal{U} \cong \mathcal{O}_B(D)$ for some effective divisor D on B and $\mathcal{L} \cong \mathcal{L}_1 \otimes \pi^*\mathcal{O}_B(-D)$ as required. $\boxed{2.4.2}$

Lemma 2.4.3. *Suppose $\mathcal{E} \in SL_0(Y, B)$ restricts to a direct sum of half-periods on a generic fibre of π, then there exists a line bundle $\mathcal{L}_1 \in \mathrm{Pic}^0(Y, B)$ such that $\pi_*(\mathcal{L}_1^{-1}\mathcal{E})) \cong \mathcal{W}$, where $\mathcal{W}$ is a rank 2 bundle on B with the property that $h^0(B, \mathcal{W}) \neq 0$, but for any line bundle $\mathcal{U}$ on B with $\deg \mathcal{U} < 0$, the space $H^0(B, \mathcal{U}\mathcal{W}) = 0$. If $\mathcal{L}$ is a line bundle which admits a non-trivial map to $\mathcal{E}$, then it is isomorphic to an element of the set*

$$\{\mathcal{L}_1 \otimes \pi^*\mathcal{V} : \mathcal{V} \text{ a line bundle on } B \text{ with } \deg \mathcal{V} \leq 0\}.$$

Proof of 2.4.3: The graph of $\mathcal{E}$ must be that of a constant map, and so by (2.2.8) there exists a line bundle $\mathcal{L}_0 \in \mathrm{Pic}^0(Y, B)$ whose graph maps to this constant under the quotient map $T \to \mathbf{P}^1$. The hypotheses imply that $\pi_*(\mathcal{L}_0^{-1}\mathcal{E})$ is a torsion-free rank 2 sheaf on B, and so corresponds to a rank 2 vector bundle, $\mathcal{W}'$ say. By [32, p.372], there exists a line bundle $\mathcal{U}$ on B such that if we define $\mathcal{W} = \mathcal{W}'\mathcal{U}$, then $\mathcal{W}$ satisfies the stated property. Hence we may take $\mathcal{L}_1 = \mathcal{L}_0 \otimes \pi^*\mathcal{U}$.

If $\mathcal{L}$ is a line bundle admitting a non-trivial map to $\mathcal{E}$, then $\mathcal{L} \cong \mathcal{L}_1 \otimes \pi^*\mathcal{V}$ for some line bundle $\mathcal{V}$ on B and

$$\pi_*(\mathcal{L}^{-1}\mathcal{E}) \cong \pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \otimes \mathcal{V}^{-1} \cong \mathcal{V}^{-1}.$$

Since $h^0(B; \pi_*(\mathcal{L}^{-1}\mathcal{E})) \neq 0$, we must have $\deg \mathcal{V} \leq 0$. $\boxed{2.4.3}$

Definition 2.4.4. If $\mathcal{E} \in SL_0(Y, B)$ has a graph of the form (2.4.1) for some holomorphic map $\rho : B \to T$, we say that $\mathcal{E}$ *splits*.

Definition 2.4.5. If $\mathcal{E} \in SL_0(Y, B)$ splits, the line bundles $\mathcal{L}_1, \mathcal{L}_2$ constructed in (2.4.2) and (2.4.3) are called the *maximal destabilising line bundles* of $\mathcal{E}$.

Suppose now that $\mathcal{E} \in SL_0^k(Y)$ with $k > 0$ satisfies the hypotheses of (2.4.2). We then have

$$h^1(\pi^{-1}(x); \mathcal{L}_1^{-1}\mathcal{E}) = 1, \text{ for generic } x \in B,$$

with exceptions only when $\mathcal{E}|_{\pi^{-1}(x)}$ is of type (3) or a direct sum of half-periods. Hence $R^1\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \mathcal{V}^{-1} \oplus \mathcal{S}$, where $\mathcal{V}$ is a line bundle on B and $\mathcal{S}$ is a skyscraper sheaf supported at the exceptional points. The dual sheaf $\{R^1\pi_*(\mathcal{L}^{-1}\mathcal{E})\}^* = \mathcal{H}om_{\mathcal{O}_Y}(R^1\pi_*(\mathcal{L}^{-1}\mathcal{E}), \mathcal{O}_Y)$ is the locally free rank 1 sheaf $\mathcal{V}$.

The bundles $\mathcal{L}_1, \mathcal{L}_2$ are not independent. In this case we have:

Lemma 2.4.6. *If $\mathcal{E} \in SL_0^k(Y)$ with $k > 0$ satisfies the hypotheses of (2.4.2), then $\mathcal{L}_1$ and $\mathcal{L}_2$ are related by*

$$\mathcal{L}_1\mathcal{L}_2 \cong \pi^*\{R^1\pi_*(\mathcal{L}_1^{-1}\mathcal{E})\}^*.$$

Proof of 2.4.6: Since $\pi : Y \to B$ is principal, its relative canonical bundle is trivial and hence, by relative Serre duality [36], [12, p.89], $\pi_*(\mathcal{L}_1\mathcal{E}) \cong \{R^1\pi_*(\mathcal{L}_1^{-1}\mathcal{E})\}^* = \mathcal{V}$. Thus $\pi_*((\mathcal{L}_1^{-1}\mathcal{V})^{-1}\mathcal{E}) \cong \mathcal{O}_B$, and the results follows by uniqueness. $\boxed{2.4.6}$

Given $\mathcal{E} \in SL_0(Y)$ with a maximal destabilising line bundle $\mathcal{L}$, let $s \in H^0(Y; \mathcal{L}^{-1}\mathcal{E})$ be a non-zero section. If s vanishes on a fibre $f = \pi^{-1}(x)$ and $\mathcal{E}|_f$ is of type (1) or (2), then s must vanish along the entire fibre. We then obtain a non-trivial map $\mathcal{L} \otimes \pi^*\mathcal{O}_B(x) \to \mathcal{E}$, contradicting the maximality of $\mathcal{L}$. Hence the zero-set of s must be contained within fibres of type (3). If $f = \pi^{-1}(x)$ is such a fibre and s vanishes on all of f, again we obtain a non-trivial map $\mathcal{L} \otimes \pi^*\mathcal{O}_B(x) \to \mathcal{E}$. Thus s can only vanish on a 0-dimensional subscheme. Hence we have shown:

Lemma 2.4.7. *If $\mathcal{E} \in SL_0(Y)$ splits and has a maximal destabilising line bundle $\mathcal{L}$ then any section of $\mathcal{L}^{-1}\mathcal{E}$ vanishes on a 0-dimensional subscheme of Y.*

This situation is well-known and is described in detail in [30, p.726] and [39, p.90]. Here we summarise the details. Let Y be a complex manifold (of any dimension) and $\mathcal{E} \to Y$ a holomorphic SL(2, C) bundle. Suppose $\mathcal{L}$ is a line bundle on Y such that the bundle $\mathcal{F} = \mathcal{L}^{-1}\mathcal{E}$ admits a non-trivial section s vanishing on a codimension 2 subscheme of Y. On any holomorphic chart U of Y where there exists a local holomorphic frame $\{e_1, e_2\}$ for $\mathcal{F}$, the section s restricted to U may be written

$$s|_U = s_1e_1 + s_2e_2,$$

for some $s_1, s_2 \in \mathcal{O}_U$. The zero-set Z is then given locally as the set of $x \in U$ such that $s_1(x) = s_2(x) = 0$, and so is a locally complete intersection. Its ideal sheaf $\mathcal{J}_Z$ is given on U by the ideal $(s_1, s_2)\mathcal{O}_U$ of $\mathcal{O}_U$. If we denote by $\{\eta_1, \eta_2\}$ the dual frame to $\{e_1, e_2\}$ for the bundle $\mathcal{F}^*$ over U, then we have a local resolution for $\mathcal{J}_Z$ on U

$$0 \longrightarrow \Lambda^2\mathcal{F}^*|_U \xrightarrow{\alpha} \mathcal{F}^*|_U \xrightarrow{\beta} \mathcal{J}_Z|_U \longrightarrow 0,$$

where α and β are defined by

$$\alpha(\eta_1 \wedge \eta_2) = \eta_1(s)\eta_2 - \eta_2(s)\eta_1; \qquad \textbf{2.4.8.}$$

$$\beta(f\eta_1 + g\eta_2) = f\eta_1(s) + g\eta_2(s), \qquad \textbf{2.4.9.}$$

for $f, g \in \mathcal{O}_U$. The local resolutions fit together to give the Koszul complex of Z:

2.4.10. $$0 \longrightarrow \Lambda^2\mathcal{F}^* \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{J}_Z \longrightarrow 0.$$

Tensoring (2.4.10) with $\det\mathcal{F}$ and using the natural isomorphism $\mathcal{F}^* \otimes \det\mathcal{F} \cong \mathcal{F}$ gives an exact sequence of sheaves

2.4.11. $$0 \longrightarrow \mathcal{O}_X \xrightarrow{s_*} \mathcal{F} \longrightarrow \det\mathcal{F} \otimes \mathcal{J}_Z \longrightarrow 0.$$

Replacing $\mathcal{F}$ with $\mathcal{L}^{-1}\mathcal{E}$, the sequence becomes

2.4.12. $$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0.$$

We denote the structure sheaf $\mathcal{O}_X/\mathcal{J}_Z$ of Z by $\mathcal{O}_Z$.

Returning now to the case where $Y \to B$ is a non-trivial principal elliptic surface we obtain:

Proposition 2.4.13. *If $\pi : Y \to B$ is a non-trivial principal elliptic surface, then any bundle $\mathcal{E} \in SL_0(Y,B)$ which splits with maximal destabilising line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ can be written in the form of an extension of sheaves*

2.4.14. $$0 \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_i^{-1}\mathcal{J}_{Z_i} \longrightarrow 0, \ \textit{for } i = 1,2,$$

where $(Z_i, \mathcal{O}_{Z_i})$ is a locally complete codimension 2 intersection. The zero schemes $(Z_i, \mathcal{O}_{Z_i})$ are parameterised by $\mathbf{P}H^0(Y;\mathcal{L}_i^{-1}\mathcal{E})$.

Proof of 2.4.14: The last statement follows since each Z_i is the zero-scheme of a non-trivial section of $H^0(Y,\mathcal{L}_i^{-1}\mathcal{E})$, and so does not change if we multiply the section by a non-zero scalar. $\boxed{2.4.14}$

Following Tyurin [43] we shall call a codimension 2 locally complete intersection $(Z,\mathcal{O}_Z)$ on a surface Y a *cluster*, and denote it simply by Z.

Lemma 2.4.15. *If Y is a compact complex surface and $\mathcal{E}$ is a rank 2 locally free sheaf given as an extension as in (2.4.12) for some line bundle $\mathcal{L}$ and cluster Z on Y, then*

$$c_2(\mathcal{E}) = l(Z) - c_1(\mathcal{L})^2.$$

Proof of 2.4.15: Apply the Chern character formula to (2.4.12) and the structure sequence of Z:

$$0 \longrightarrow \mathcal{J}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

$\boxed{2.4.15}$

2.5. Stable SL(2, C) bundles on Hopf surfaces

In this section we describe how the results of the previous sections specialise to the Hopf surface $\pi : X \to \mathbf{P}^1$. See also [16].

2.5.1. Line bundles on Hopf surfaces

Since X is topologically $S^1 \times S^3$ we have $H^2(X;\mathbf{Z}) = 0$, and by the Riemann-Roch theorem $h^{0,1}(X) = 1$. Hence $\mathrm{Pic}^0(X,\mathbf{P}^1) = \mathrm{Pic}^0(X) = \mathrm{Pic}(X)$ where:

$$\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*) \cong H^1(X,\mathcal{O}_X)/H^1(X;\mathbf{Z}) \cong \mathbf{C}/\mathbf{Z} \cong \mathbf{C}^*.$$

As in (2.2.15) we may give an explicit representative $\mathcal{L}_\mu$ of the class in $\mathrm{Pic}(X)$ represented by $\mu \in \mathbf{C}^*$ by defining $\mathcal{L}_\mu$ to be the quotient of $\mathbf{H}^* \times \mathbf{C}$ (with its given complex structure) by the $\mathbf{Z}$ action generated by

$$(z,\xi) \mapsto (\lambda z, \mu\xi).$$

Clearly on the fibres T of π, the line bundle $\mathcal{L}_\mu$ restricts to the bundle $\mathcal{V}_\mu \in \mathrm{Pic}^0(T) \cong T$ defined in 2.1. Since all holomorphic maps from $\mathbf{P}^1$ to T are constant, the exact sequences (2.2.10) and (2.2.12) are identical, and (2.2.12) becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(\mathbf{P}^1) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}^0(T) & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\pi^*} & \mathbf{C}^* & \xrightarrow{\rho} & T & \longrightarrow & 1
\end{array}$$

where $\pi^*(n) = \lambda^n$ and $\rho(\mu) = \langle\mu\rangle \in \mathbf{C}^*/\mathbf{Z}$.

Denote the pull-back of $\mathcal{O}_{\mathbf{P}^1}(n)$ by $\mathcal{O}_X(n)$ for $n \in \mathbf{Z}$. By Kodaira's canonical bundle formula [12], we have $\mathcal{K}_X \cong \mathcal{O}_X(-2)$. Using the original quotient construction of X, one sees that the tangent bundle of X is isomorphic to $\mathcal{L}_\lambda \oplus \mathcal{L}_\lambda$. Hence $\mathcal{L}_\lambda^2 \cong \mathcal{O}_X(2)$. Since $\mathcal{L}_\lambda$ is trivial on fibres of π, there is only one possible choice of square-root, namely $\mathcal{L}_\lambda \cong \mathcal{O}_X(1)$.

Using the fact that $\mathrm{Todd}(X) = 1$, an application of Riemann-Roch to any line bundle $\mathcal{L}$ on X gives $\chi(X;\mathcal{L}) = 0$. Thus, by Serre duality

$$h^1(\mathcal{L}) = h^0(\mathcal{L}) + h^2(\mathcal{L}) = h^0(\mathcal{L}) + h^0(\mathcal{L}^{-1} \otimes \mathcal{K}_X).$$

If a line bundle $\mathcal{L}$ on X admits a non-trivial section, then on restriction to the fibres of π that section must be constant, so $\mathcal{L}$ is the pull-back of a bundle on $\mathbf{P}^1$; i.e., $\mathcal{L} \cong \mathcal{O}_X(n)$ for some $n \in \mathbf{Z}$ with $n \geq 0$. Thus, if $\mathcal{L}$ is not of the form $\mathcal{O}_X(n)$ for any $n \in \mathbf{Z}$, we have

$$h^i(\mathcal{L}) = 0, \quad \text{for } i = 0,1,2.$$

Otherwise, if $\mathcal{L} \cong \mathcal{O}_X(m)$ for $m \in \mathbf{Z}$, we have $h^0(\mathcal{L}) = h^0(\mathcal{O}_{\mathbf{P}^1}(m))$ and so we obtain:

$$h^0(\mathcal{O}_X(m)) = \begin{cases} m+1, & \text{for } m \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and so

$$h^1(\mathcal{O}_X(m)) = h^0(\mathcal{O}_X(m)) + h^0(\mathcal{O}_X(2-m)).$$

Define a metric $||\cdot||$ on the trivial bundle over $\mathbf{C}^*$ by

2.5.1. $$||s(z)||^2 = |s(z)|^2(|z_0|^2 + |z_1|^2)^{-\log|\mu|/\log\lambda},$$

where $|\cdot|$ is the usual L^2-metric on $\mathbf{C}$ and s is a section. This metric is then invariant under the action of $\mathbf{Z}$ defining $\mathcal{L}_\mu$ and so descends to give a metric on $\mathcal{L}_\mu$, whose curvature form F is given by

$$F = -\frac{\log|\mu|}{\log\lambda}\bar{\partial}\partial\log(|z_0|^2 + |z_1|^2).$$

Integrating F against ω and choosing the constant κ in (1.4.1) appropriately, we have

$$\deg \mathcal{L}_\mu = \log|\mu|/\log\lambda,$$

and in particular $\deg \mathcal{O}_X(n) = n$ for $n \in \mathbf{Z}$.

2.5.2. SL(2, C) bundles on Hopf surfaces

If $\mathcal{E} \to X$ is an SL(2, C) bundle with $c_2(\mathcal{E}) = k > 0$, then the fact that $\mathcal{E} \in SL_0(X)$ follows from the following easy generalisation of Proposition 3.2.2. of [16]. The result is stated here both for completeness and because we shall need it in this generality in Chapter 6.

Note that when $B = \mathbf{P}^1$, by (2.2.1) the group $H^2(Y;\mathbf{Z})$ must be pure torsion whenever Y is non-trivial.

Lemma 2.5.2 ([16], Prop.3.2.2). *Suppose $\pi : Y \to \mathbf{P}^1$ is a non-trivial principal elliptic fibration then every holomorphic* SL(2, C) *bundle $\mathcal{E}$ on Y with $c_2(\mathcal{E}) > 0$ belongs to $SL_0(Y)$.*

Proof of 2.5.2: Let $\mathcal{E}$ be as in the statement. If $\mathcal{E}$ is of type (3) on a generic and hence, by semi-continuity, every fibre of π then we define a coherent subsheaf S of $\mathcal{E}$ as follows. For any line bundle $\mathcal{L}$ on Y the hypotheses give $h^0(\pi^{-1}(x);\mathcal{L}\mathcal{E}) > 0$ for all $x \in \mathbf{P}^1$. The direct image sheaf $\pi_*(\mathcal{L}\mathcal{E})$ is torsion-free and so defines a vector bundle on $\mathbf{P}^1$. We have a natural map $\pi^*\pi_*(\mathcal{L}\mathcal{E}) \to \mathcal{L}\mathcal{E}$ which together with the natural evaluation map $\mathcal{L}\mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{E}$ defines a map $(\pi^*\pi_*(\mathcal{L}\mathcal{E})) \otimes \mathcal{L}^{-1} \to \mathcal{E}$. Choose $\mathcal{L}$ so that $\mathcal{L} \not\cong \pi^*\mathcal{O}_{\mathbf{P}^1}(n)$ for any $n \in \mathbf{Z}$, and let S be the image of $(\pi^*\pi_*(\mathcal{L}\mathcal{E}) \otimes \mathcal{L}^{-1}) \oplus \pi^*\pi_*\mathcal{E}$ under the evaluation maps. Since $H^2(Y;\mathbf{Z})$ is pure torsion, the first Chern class $c_1(S)$ must vanish on restriction to fibres of π. However, on each fibre T of π, the restriction of the sheaf S is isomorphic to the positive sub-line bundle of $\mathcal{E}$ generated by the image under the evaluation map of those sections of $\mathcal{E}|_T$ and $\mathcal{L}\mathcal{E}|_T$ which extend to a neighbourhood of the fibre. Any such section must vanish at some point of T, hence $c_1(S|_T) \neq 0$, giving a contradiction. $\boxed{2.5.2}$

The divisor $D(\mathcal{E})$ of $\mathcal{E}$ is therefore well-defined and is an element of a linear system $|\mathcal{O}(p,q)|$ on $\mathbf{P}^1 \times \mathbf{P}^1$, for some $p, q \in \mathbf{Z}$. By intersecting $D(\mathcal{E})$ with a generic horizontal and vertical fibre we see that $p = k$ and $q = 1$, so $D(\mathcal{E})$ is a $(k,1)$ divisor on the quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$. As in (2.3.9), we may write $D(\mathcal{E})$ uniquely in the form;

2.5.3. $$D(\mathcal{E}) \cong G_\alpha + \sum_{j=1}^{s} n_j F_j$$

where G_α is the graph of a degree r rational map $\alpha : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$, for some r with $0 \leq r \leq k$, and the $F_j = \{x_j\} \times \mathbf{P}^1$ are the vertical fibres over a set $\{x_1, \ldots, x_s\}$ of *distinct* points in $\mathbf{P}^1$ for which $\mathcal{E}\,|_{\pi^{-1}(x_j)}$ is of type (3).

2.5.3. Stratification of the moduli space

For $k > 0$ let $\mathcal{M}_k$ denote the moduli space of stable SL(2, C) bundles $\mathcal{E}$ on X with $c_2(\mathcal{E}) = k$. Taking the divisor of a bundle defines a holomorphic map

$$D : \mathcal{M}_k \longrightarrow \mathbf{P}H^0(\mathcal{O}(k,1)) \cong \mathbf{P}^{2k+1}. \tag{2.5.4}$$

The decomposition (2.5.3), gives a stratification of $\mathcal{M}_k$ into disjoint subspaces $\mathcal{M}_k^r$ for $r = 0, 1, \ldots, k$, where $\mathcal{M}_k^r$ denotes the subset of $\mathcal{M}_k$ consisting of elements $\mathcal{E}$ whose graph G_α in (2.5.3) is that of a degree r rational map $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$.

Remark 2.5.5. Note that if $\mathcal{E} \in \mathcal{M}_k^r$ for $r > 0$, then there can be no non-trivial map from a line bundle on X to $\mathcal{E}$ and consequently $\mathcal{E}$ must be stable.

In the following chapters we shall investigate the structure of the various strata and the map D of (2.5.4). It will turn out that the stratification is rather coarse and as k increases it rapidly becomes more complicated to give an explicit description of the various strata.

Chapter 3

The lowest stratum

In this chapter we study SL(2, C) bundles $\mathcal{E}$ on X with $c_2(\mathcal{E}) = k > 0$, whose isomorphism class defines an element of the "lowest" or 0^{th} stratum $\mathcal{M}_k^0$ of $\mathcal{M}_k$. These are precisely the bundles which when twisted by some line bundle $\mathcal{L}$ admit a non-trivial global section. A construction of Serre allows us to reconstruct the bundle given $\mathcal{L}$ and the zero-scheme of any such section. We determine when this construction results in stable bundles, and calculate the dimension of the stratum.

This method is purely algebraic in character and avoids the need for the gluing employed by Braam and Hurtubise. Moreover, the rôle played by configurations of points on X becomes somewhat clearer. When $k = 1$, the construction becomes especially simple, allowing us to give a complete description of the stratum.

3.1. Destabilising line bundles

We consider the 'lowest' stratum $\mathcal{M}_k^0$ consisting of those elements of $\mathcal{M}_k$ whose divisors are of the form

3.1.1. $$\mathbf{P}^1 \times \{l\} + \sum\nolimits_{j=1}^{r} n_j F_j,$$

where $l \in \mathbf{P}^1 \cong \mathrm{Pic}^0(T)/\mathbf{Z}_2$. These are precisely the bundles which admit a non-trivial map from a line bundle on X.

Below we summarise some of the results of Chapter 2 as they apply to Hopf surfaces. The first result was obtained by Braam and Hurtubise in [16].

Proposition 3.1.2. *Suppose $\mathcal{E} \in \mathcal{M}_k$ has a divisor of the form (3.1.1), then there exist two line bundles, $\mathcal{L}_1, \mathcal{L}_2$, such that any bundle admitting a non-trivial map to $\mathcal{E}$ is isomorphic to one in the set:*

$$\{\mathcal{L}_1(-n), \mathcal{L}_2(-n) : n \in \mathbf{Z}, n \geq 0\}.$$

If $\mathcal{E}$ restricts to an extension of half-periods on a generic fibre, then $\mathcal{L}_1 = \mathcal{L}_2$ and $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \mathcal{O}_{\mathbf{P}^1}$ or $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m)$ for some $m \in \mathbf{Z}$, with $m \leq 0$ according *to whether the extension is generically of type (1) or (2). Otherwise we have* $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \pi_*(\mathcal{L}_2^{-1}\mathcal{E}) \cong \mathcal{O}_{\mathbf{P}^1}$.

Proof of 3.1.2: This follows by (2.4.2) and (2.4.3). In the case where $B = \mathbf{P}^1$, the rank 2 bundle $\mathcal{W}$ of (2.4.3) must be a direct sum of the form $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m)$ with $m \geq 0$ by the Birkhoff-Grothendieck theorem [32, p.376]. $\boxed{3.1.2}$

Using (1.5.1), we obtain:

Proposition 3.1.3. *A bundle $\mathcal{E} \in \mathcal{M}_k^0$ is stable if and only if its maximal destabilising line bundles, $\mathcal{L}_1, \mathcal{L}_2$, satisfy*

$$\deg \mathcal{L}_1 < 0 \text{ and } \deg \mathcal{L}_2 < 0.$$

The bundles $\mathcal{E} \in \mathcal{M}_k^0$ which admit more than a $\mathbf{P}^1$'s worth of maps from their maximal destabilising line bundle form an exceptional class. We therefore make the following definition:

Definition 3.1.4. Let Σ_k denote the subset of $\mathcal{M}_k^0$ consisting of isomorphism classes of bundles $\mathcal{E}$ with maximal destabilising line bundle $\mathcal{L}$ satisfying $\pi_*(\mathcal{L}^{-1}\mathcal{E}) \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$

Proposition 3.1.5. *Any bundle $\mathcal{E} \in \mathcal{M}_k^0$ with maximal destabilising line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ can be written in the form of an extension of* ***sheaves****:*

3.1.6. $$0 \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_i^{-1}\mathcal{J}_{Z_i} \longrightarrow 0, \text{ for } i = 1, 2,$$

where Z_i is a cluster on X. The clusters Z_i are uniquely determined unless $\mathcal{E} \in \Sigma_k$, in which case there is a family of such schemes parameterised by $\mathbf{P}^1$.

Proof of 3.1.5: This is a special case of (2.4.13) using the fact that if $\mathcal{E} \in \mathcal{M}_k^0 \setminus \Sigma_k$ then $h^0(\mathcal{L}_i^{-1}\mathcal{E}) = 1$ for $i = 1, 2$, and if $\mathcal{E} \in \Sigma_k$, we have $h^0(\mathcal{L}_i^{-1}\mathcal{E}) = 2$. $\boxed{3.1.5}$

Lemma 3.1.7. *If $\mathcal{E} \in \mathcal{M}_k^0$ is given as an extension of sheaves as in (2.4.12) with $\mathcal{L}$ a line bundle and Z a cluster on X, then $c_2(\mathcal{E}) = l(Z)$.*

Proof of 3.1.7: This follows from (2.4.15) using the fact that $H^2(X; \mathbf{Z}) = 0$. $\boxed{3.1.7}$

3.2. Relationship with the divisor

Suppose $\mathcal{E}$ is an SL(2, **C**) bundle on X which is expressed as an extension as in (2.4.12). Denote the support of Z by Supp$Z = \{x_1, \ldots, x_l\}$. On a fibre f of $\pi : X \to \mathbf{P}^1$ which does not contain a point of Supp$Z$, the exact sequence (2.4.12) restricts to

$$0 \longrightarrow \mathcal{L}|_f \longrightarrow \mathcal{E}|_f \longrightarrow \mathcal{L}^{-1}|_f \longrightarrow 0$$

so $\mathcal{E}|_f$ is of type (1) or (2), and its divisor contains the graph of the constant map $\{\mathcal{L}\}$, where $\{\mathcal{L}\}$ is the image of $\mathcal{L} \in \mathrm{Pic}(X)$ in $\mathbf{P}^1$ under the quotient map.

If, however, we take a fibre f which does meet Supp$Z$, then $\mathcal{J}_Z|_f$ is the ideal sheaf of a divisor Z_f on f, namely the zero-set of $s|_f$. Note that $l(Z_f) \leq \sum_{x_i \in f} l(Z_{x_i})$ and the inequality is strict if at some point x_i, the section s vanishes to order greater than 1 in a direction transverse to f. In any case, the exact sequence restricts to

$$0 \longrightarrow \mathcal{L}|_f \otimes \mathcal{O}_f(Z_f) \longrightarrow \mathcal{E}|_f \longrightarrow \mathcal{L}^{-1}|_f \otimes \mathcal{O}_f(-Z_f) \longrightarrow 0.$$

Since $H^1(\mathcal{O}_f(2Z_f)) = 0$, this sequence splits and

$$\mathcal{E}|_f \cong \{\mathcal{L}|_f \otimes \mathcal{O}_f(Z_f)\} \oplus \{\mathcal{L}^{-1}|_f \otimes \mathcal{O}_f(-Z_f)\}.$$

Hence the fibres of X on which $\mathcal{E}$ is of type (3) are precisely those containing Supp$Z$, and the degrees of the bundles involved in the splitting of $\mathcal{E}$ on such fibres are determined by the behaviour of $\mathcal{J}_Z$, or equivalently $\mathcal{O}_Z$, on restriction.

If $\mathcal{L}'$ denotes a bundle on X such that $\mathcal{L}'|_f$ is not isomorphic to either $\mathcal{L}|_f$ or $\mathcal{L}^{-1}|_f$ for a generic (and hence every) fibre f of π, then applying the right derived functor $R\pi_*$ (see [32]) to the exact sequence

$$0 \longrightarrow \mathcal{L}'\mathcal{L} \longrightarrow \mathcal{L}'\mathcal{E} \longrightarrow \mathcal{L}'\mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0,$$

gives an isomorphism

$$R^1\pi_*(\mathcal{L}'\mathcal{E}) \cong R^1\pi_*(\mathcal{L}'\mathcal{L}^{-1}\mathcal{J}_Z).$$

The skyscraper sheaf $R^1\pi_*(\mathcal{L}'\mathcal{E})$ is supported on a set of distinct points $\{p_1, \ldots, p_r\}$ in $\mathbf{P}^1$. Applying the same functor to the structure sequence

$$0 \rightarrow \mathcal{L}'\mathcal{L}^{-1}\mathcal{J}_Z \rightarrow \mathcal{L}'\mathcal{L}^{-1} \rightarrow \mathcal{L}'\mathcal{L}^{-1}\mathcal{O}_Z \rightarrow 0,$$

and using the fact that $\mathcal{L}'\mathcal{L}^{-1}$ is non-trivial on restriction to the fibres of π in the resulting long exact sequence of direct image sheaves gives an isomorphism

$$R^1\pi_*(\mathcal{L}'\mathcal{L}^{-1}\mathcal{J}_Z) \cong \pi_*(\mathcal{L}'\mathcal{L}^{-1}\mathcal{O}_Z).$$

The latter sheaf is supported on $\pi(\mathrm{Supp}Z)$ and for a point $p_j \in \pi(\mathrm{Supp}Z)$, we have

$$(\pi_*(\mathcal{L}'\mathcal{L}^{-1}\mathcal{O}_Z))_{p_j} \cong \sum_{y \in \mathrm{Supp}Z \cap \pi^{-1}(p_j)} (\mathcal{O}_Z)_y.$$

Hence we see the multiplicity n_j of the vertical fibre in the divisor of $\mathcal{E}$ above p_j is given by

3.2.1.
$$n_j = \sum_{x \in \mathrm{Supp}Z \cap \pi^{-1}(p_j)} l(Z_x).$$

Hence, the divisor of any bundle $\mathcal{E} \in \mathcal{M}_k^0$, which is expressed as an extension of the form (2.4.12), is given by

$$\mathbf{P}^1 \times \{\mathcal{L}\} + \sum_{j=1}^{r} n_j F_j,$$

where $\mathrm{Supp}(\pi_* Z) = \{p_1, \ldots, p_r\}$, $F_j = \{p_j\} \times \mathbf{P}^1$, and the multiplicity n_j is given by (3.2.1).

We shall have need of the following results later:

Lemma 3.2.2. *If $\mathcal{E}$ represents a class in Σ_k, then on a fibre F of π where $\mathcal{E}$ is of type (3)*

$$\mathcal{E}|_F \cong \mathcal{L}_r \oplus \mathcal{L}_r^{-1},$$

for some $\mathcal{L}_r \in \mathrm{Pic}^r(F)$ with $r \geq 2$.

Proof of 3.2.2: If $\mathcal{E} \in \Sigma_k$ and $\mathcal{L}$ is a maximal destabilising line bundle, then $h^0(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) = 2$ for generic $x \in \mathbf{P}^1$. Hence, by semi-continuity, $h^0(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) \geq 2$ for all $x \in \mathbf{P}^1$ and the splitting type on fibres where $\mathcal{E}$ is of type (3) is as stated. $\boxed{3.2.2}$

Corollary 3.2.3. *If $\mathcal{E} \in \Sigma_k$, then the number of fibres N where $\mathcal{E}$ is of type (3) satisfies*

$$1 \leq N \leq \frac{k}{2}.$$

In particular, $\Sigma_1 = \emptyset$.

3.3. The Serre construction

We have seen that the bundles under consideration are all sheaf extensions of the form

3.3.1. $$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0,$$

for some line bundle $\mathcal{L}$ and cluster Z on X. We would like to use such extensions to reconstruct the bundle given the data $\mathcal{L}$ and Z. The results of this section are standard (see [30, p.726], [32, Chapter 3], [39, p.90] and [40]), and for most of it we shall drop the assumption that X be a surface, assuming only that Z is a codimension 2 locally complete intersection subscheme on X. Restricting the exact sequence (3.3.1) to Z gives an isomorphism $\mathcal{L}\mathcal{E}|_Z \cong \mathcal{N}_Z^*$, where $\mathcal{N}_Z^* = \mathcal{J}_Z/\mathcal{J}_Z^2$ is the conormal sheaf of Z in X. Hence a necessary condition for such an extension to exist is that $\mathcal{L}^2|_Z$ be isomorphic to $(\det \mathcal{N}_Z)^*$. This condition also turns out to be sufficient as we now explain.

Tensoring the structure sequence of Z,

$$0 \longrightarrow \mathcal{J}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

by $\mathcal{L}^{-1}$ and taking $\mathrm{Ext}^{*}(\cdot,\mathcal{L})$ gives a long exact sequence:

$$3.3.2. \qquad \begin{array}{lllllll} \to & \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) & \to & H^1(\mathcal{L}^2) & \to & \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}) & \to \\ & \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) & \to & H^2(\mathcal{L}^2) & \to & \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}) & \to . \end{array}$$

Since Z is a codimension 2 locally complete intersection, there are isomorphisms of $\mathcal{O}_X$-modules:

$$3.3.3. \qquad \begin{array}{rcl} \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z,\mathcal{F}) & = & 0\ ,(r \neq 2); \\ \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_Z,\mathcal{F}) & \cong & \mathcal{H}om_{\mathcal{O}_Z}(\det \mathcal{N}_Z^*, \mathcal{F}\mathcal{O}_Z) \end{array}$$

for any locally free sheaf $\mathcal{F}$ on X [2, Theorem 4.5]. Hence if $\mathcal{L}$ is a line bundle on X with $\mathcal{L}^2|_Z \cong (\det \mathcal{N}_Z)^*$ we have an isomorphism

$$\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_Z,\mathcal{L}^2) \cong \mathcal{O}_Z.$$

Using the Ext spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}),$$

we obtain

$$3.3.4. \qquad \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) \cong H^0(\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L})) \cong H^0(\mathcal{O}_Z).$$

Furthermore, by (3.3.3) the terms $E_2^{1,0}$ and $E_2^{0,1}$ of the spectral sequence both vanish giving

$$\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) = 0.$$

Substituting this in the long exact sequence (3.3.2) gives the *fundamental exact sequence:*

$$3.3.5. \qquad \begin{array}{rllllll} 0 & \to & H^1(\mathcal{L}^2) & \to & \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}) & \xrightarrow{\rho} \\ H^0(\mathcal{O}_Z) & \to & H^2(\mathcal{L}^2) & \to & \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}) & \to & . \end{array}$$

Taking $\mathcal{E}xt^{*}(\cdot,\mathcal{L})$ of the exact sequence

$$0 \to \mathcal{L}^{-1}\mathcal{J}_Z \to \mathcal{L}^{-1} \to \mathcal{L}^{-1}\mathcal{O}_Z \to 0,$$

results in an isomorphism of sheaves,

$$3.3.6. \qquad \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) \cong \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}),$$

and a corresponding isomorphism $H^0(\mathcal{O}_Z) \cong H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}))$. The map $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z,\mathcal{L}))$ induced by the composition of this isomorphism with the map ρ in the fundamental exact sequence is just the natural localisation morphism.

When X is a *surface*, we have $\mathrm{Ext}^3(\mathcal{L}^{-1}\mathcal{O}_Z,\mathcal{L}) = 0$. This and an application of Grothendieck-Serre duality [30, p.707 – 708] to the final three terms of (3.3.5) give the *fundamental exact sequence for surfaces*:

$$0 \to H^1(\mathcal{L}^2) \longrightarrow \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \xrightarrow{\rho}$$

3.3.7.
$$\begin{array}{ccccc} H^0(\mathcal{O}_Z) & \xrightarrow{\delta} & H^2(\mathcal{L}^2) & \to & \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \to 0 \\ \| & & \| & & \| \\ H^0(\mathcal{L}^{-2}\mathcal{K}_X\mathcal{O}_Z)^* & \xrightarrow{q^T} & H^0(\mathcal{L}^{-2}\mathcal{K}_X)^* & \xrightarrow{\iota^T} & H^0(\mathcal{L}^{-2}\mathcal{K}_X\mathcal{J}_Z)^*, \end{array}$$

where q^T and ι^T are the transposes of the natural maps q, ι in the exact sequence

$$0 \longrightarrow H^0(\mathcal{L}^{-2}\mathcal{K}_X\mathcal{J}_Z) \xrightarrow{\iota} H^0(\mathcal{L}^{-2}\mathcal{K}_X) \xrightarrow{q} H^0(\mathcal{L}^{-2}\mathcal{K}_X\mathcal{O}_Z).$$

Extensions of type (3.3.1) are classified by $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$. If η is a class in $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$, then $\rho(\eta)$ is an element of $H^0(\mathcal{O}_Z)$. We are now ready to state a fundamental result attributed in the literature to Serre:

Proposition 3.3.8 (Serre). *If $\mathcal{L}$ satisfies $\mathcal{L}^2|_Z \cong (\det \mathcal{N}_Z)^*$, then the rank 2 sheaf $\mathcal{E}$ defined as in (3.3.1) by an extension class $\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ is locally free if and only for each point x in the support of Z, the localisation η_x of η at x generates $(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))_x$ as an $\mathcal{O}_{X,x}$-module.*

Using the natural isomorphism (3.3.6) we then have:

Corollary 3.3.9. *A class $\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ defines a locally free sheaf $\mathcal{E}$ if and only if for all points x in the support of Z the localisation $(\rho(\eta))_x$ of $\rho(\eta) \in H^0(\mathcal{O}_Z)$ generates $\mathcal{O}_{Z,x}$ as an $\mathcal{O}_{X,x}$-module.*

Hence if the element $1 \in H^0(\mathcal{O}_Z)$ lifts to $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$, we can construct a locally free extension of the form (3.3.1).

Remark 3.3.10. When X is a *surface* the subscheme Z is supported on a finite set of points so the condition $\mathcal{L}^2|_Z \cong (\det \mathcal{N}_Z)^*$ is always satisfied.

Returning now to the case of the Hopf surface, in view of the exact sequence (3.3.7) we shall consider the two cases where $\mathcal{L}$ satisfies:

$$\mathcal{L}^2 \not\cong \mathcal{O}_X(m), \text{ for any } m \in \mathbf{Z} \text{ (Generic Case);} \qquad \mathbf{3.3.11.}$$
$$\mathcal{L}^2 \cong \mathcal{O}_X(m), \text{ for some } m \in \mathbf{Z} \text{ (Exceptional Case)} \qquad \mathbf{3.3.12.}$$

separately. Note that it is a necessary condition for a locally free extension of the form (3.3.1) to be stable that $\mathcal{L}$ satisfies $\deg \mathcal{L} < 0$.

3.4. Construction of a generic element

In the generic case, $H^1(\mathcal{L}^2) = H^2(\mathcal{L}^2) = 0$ and the exact sequence (3.3.7) gives the isomorphism
$$\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \cong H^0(\mathcal{O}_Z),$$
using (3.3.6) and (3.3.3). Thus, for any global section η of $\mathcal{O}_Z$ which generates $\mathcal{O}_{Z_x}$ as an $\mathcal{O}_{X,x}$-module for each $x \in \mathrm{Supp} Z$, we have a lift to a global extension

$\tilde{\eta} \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ and the corresponding rank 2 sheaf $\mathcal{E}$ is locally free by (3.3.8).

We now come to the question of stability for locally free extensions of the form (3.3.1), where the bundle $\mathcal{L}$ satisfies (3.3.11). Roughly speaking, this depends on how the cluster Z 'sits inside X relative to the fibres of π'. Following Friedman [24], we make the following definitions:

Definition 3.4.1. Suppose $(Z, \mathcal{O}_Z)$ is a cluster on X, supported on a single point, $x \in X$, say. Given a (reduced) fibre f of $\pi : X \to \mathbf{P}^1$, define the *fibre weight of Z at x along f, $w_f(Z_x)$*, by

3.4.2. $$w_f(Z_x) = \min\{n \in \mathbf{Z} : \mathcal{O}_X(-nf) \hookrightarrow \mathcal{J}_{Z_x}\}.$$

Definition 3.4.3. Given an arbitrary cluster $(Z, \mathcal{O}_Z)$ on X, such that $\mathrm{Supp} Z = \{x_1, \ldots, x_s\}$, we define the *fibre weight of Z along f, $w_f(Z)$*, by

3.4.4. $$w_f(Z) = \max\{w_f(Z_{x_i}) : x_i \in f\}.$$

Definition 3.4.5. The *fibre weight* $w(Z)$ of a cluster Z on X is defined by:

3.4.6. $$w(Z) = \sum_{x \in \mathbf{P}^1} w_{\pi^{-1}(x)}(Z).$$

Note that $w_f(Z) > 0$ if and only if f contains a point of $\mathrm{Supp} Z$. Thus the fibre weight $w_{\pi^{-1}(x)}$ is non-zero only for points $x \in \mathrm{Supp}(\pi_* Z)$.

We have the following technical lemma:

Lemma 3.4.7. *Suppose $\mathcal{L}$ is a line bundle and $(Z, \mathcal{O}_Z)$ a cluster on X, then*

$$h^0(X; \mathcal{L}\mathcal{J}_Z) = \begin{cases} \max\{0, \deg(\mathcal{L}) + 1 - w(Z)\}, & \text{if } \mathcal{L} \cong \mathcal{O}_X(m), \\ & \text{for some } m \in \mathbf{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of 3.4.7: Suppose $h^0(X; \mathcal{L}\mathcal{J}_Z) > 0$, then $\mathcal{L} \cong \mathcal{O}_X(m)$, for some $m \in \mathbf{Z}$. Let s be a non-trivial global section of $\mathcal{L}\mathcal{J}_Z$, vanishing on an effective divisor D on X. Since s must vanish on Z and all effective divisors on X are sums of fibres of π, we obtain

$$D = \sum_{i=1}^{r} n_i F_i + R,$$

where R is an effective divisor on X, whose support does not meet the support of Z, and $\{F_i : i = 1, \ldots, r\}$ is a collection of disjoint (reduced) fibres of π such that

$$\sum_{i=1}^{r} F_i = \pi^{-1}(\mathrm{Supp}(\pi_* Z)).$$

Hence, $\mathcal{O}_X(-n_i F_i) \hookrightarrow \mathcal{J}_{Z_x}$ and so $n_i \geq w(Z_x)$ for each point $x \in \mathrm{Supp} Z \cap F_i$. This gives $n_i \geq w_{F_i}(Z)$ for $i = 1, \ldots, r$. Consequently, we have

3.4.8. $$D = \sum_{i=1}^{r} w_{F_i}(Z).F_i + R',$$

for an effective divisor R'. Conversely, given any divisor D of the form (3.4.8) on X, with R' effective, then by definition of the fibre weight there is a non-trivial map $\mathcal{O}_X(-D) \longrightarrow \mathcal{J}_Z$.

The decomposition (3.4.8) of any divisor in the linear system $|\mathcal{O}_X(\mathcal{L})\mathcal{J}_Z|$ into fixed and moving parts gives

3.4.9.
$$\begin{aligned} h^0(\mathcal{L}\mathcal{J}_Z) &= h^0(\mathcal{O}_X(R')) \\ &= \deg(\mathcal{O}_X(R')) + 1 \\ &= \deg(\mathcal{L}) - \deg(\mathcal{O}_X(\textstyle\sum_{i=1}^{r} w_{F_i}(Z).F_i)) + 1 \\ &= \deg(\mathcal{L}) + 1 - w(Z). \end{aligned}$$

$\boxed{3.4.7}$

Lemma 3.4.10. *If $\mathcal{E}$ is a locally free extension of the form (3.3.1) with $\mathcal{L}$ generic, then $\mathcal{L}$ is maximal destabilising line bundle for $\mathcal{E}$.*

Proof of 3.4.10: Suppose not, so $\mathcal{L}(1)$ admits a non-trivial map to $\mathcal{E}$ and we have an exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{L}^{-1}\mathcal{E}(-1) \longrightarrow \mathcal{L}^{-2}(-1)\mathcal{J}_Z \longrightarrow 0,$$

giving rise to an exact sequence in cohomology:

$$0 \longrightarrow H^0(\mathcal{L}^{-1}\mathcal{E}(-1)) \longrightarrow H^0(\mathcal{L}^{-2}(-1)\mathcal{J}_Z) \longrightarrow H^1(\mathcal{O}_X(-1)) = 0.$$

Hence $h^0(\mathcal{L}^{-2}(-1)\mathcal{J}_Z) > 0$. Restricting any non-zero section of $\mathcal{L}^{-2}(-1)\mathcal{J}_Z$ to a general fibre f of π shows that $\mathcal{L}^2|_f$ admits a non-trivial section. This means $\mathcal{L}^2|_f \cong \mathcal{O}_f$, so contradicting our assumption on $\mathcal{L}$. $\boxed{3.4.10}$

We can now determine the other maximal destabilising line bundle of $\mathcal{E}$.

Proposition 3.4.11. *Under the hypotheses of Lemma 3.4.10 the maximal destabilising line bundles of $\mathcal{E}$ are $\mathcal{L}$ and $\mathcal{L}^{-1}(-w(Z))$.*

Proof of 3.4.11: Let $\mathcal{L}'$ denote the maximal destabilising line bundle other than $\mathcal{L}$, then we have $\mathcal{L}' = \mathcal{L}^{-1}(-m)$ for some $m \in \mathbf{Z}$ by (2.4.6). Tensoring the exact sequence (3.3.1) by $(\mathcal{L}')^{-1}$ and taking cohomology gives:

$$0 \to H^0((\mathcal{L}')^{-1}\mathcal{L}) \to H^0((\mathcal{L}')^{-1}\mathcal{E}) \to H^0((\mathcal{L}'\mathcal{L})^{-1}\mathcal{J}_Z) \to H^1((\mathcal{L}')^{-1}\mathcal{L}),$$

or, equivalently,

$$0 \longrightarrow H^0(\mathcal{L}^2(m)) \longrightarrow H^0((\mathcal{L}')^{-1}\mathcal{E}) \longrightarrow H^0(\mathcal{O}_X(m)\mathcal{J}_Z) \longrightarrow H^1(\mathcal{L}^2(m)).$$

The first and last terms vanish by the assumption on $\mathcal{L}^2$, giving

$$H^0((\mathcal{L}')^{-1}\mathcal{E}) = H^0(\mathcal{O}_X(m)\mathcal{J}_Z).$$

If $\mathcal{L}'$ is maximal we have $h^0((\mathcal{L}')^{-1}\mathcal{E}) = 1$, for if not and $h^0((\mathcal{L}')^{-1}\mathcal{E}) \geq 2$, choosing a divisor f in the linear system $|(\mathcal{L}')^{-1}\mathcal{E}|$ which does not meet Z defines a non-trivial map $\mathcal{L}'\mathcal{O}_X(f) \longrightarrow \mathcal{E}$, contradicting the maximality of $\mathcal{L}'$. Hence, $h^0(\mathcal{O}_X(m)\mathcal{J}_Z) = 1$. By Lemma 3.4.7, $m = \deg \mathcal{O}_X(m) = w(Z)$, giving the result. $\boxed{3.4.11}$

Corollary 3.4.12. *The maximal destabilising line bundles satisfy the equation:*

3.4.13. $$\deg \mathcal{L} + \deg \mathcal{L}' = -w(Z)$$

and the bundle $\mathcal{E}$ is stable if and only if

$$-w(Z) < \deg \mathcal{L} < 0.$$

3.5. The generic substratum

We have seen that locally free extensions $\mathcal{E}$ of the form (3.3.1) are given by a non-zero element η of $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ which satisfies (3.3.8). For a fixed $\mathcal{L}$, two such extensions produce isomorphic bundles if and only if they lie in the same orbit of the action of $\mathrm{Aut}\mathcal{L} \cong \mathbf{C}^*$ on $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$. Hence the isomorphism class of $\mathcal{E}$ is determined uniquely by the image of η in $\mathbf{P}(\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})) \cong \mathbf{P}^{k-1}$. Let $\mathrm{Hilb}^k X$ denote the 'Hilbert scheme' or Douady manifold of zero-cycles of length k on X [7, p.52], [13]. (Recall $\mathrm{Hilb}^k X$ is a canonical resolution of singularities of the k-fold symmetric product of X.) The following proposition is analogous to Proposition (3.4) of [24].

Proposition 3.5.1. *The space of moduli of stable* $\mathrm{SL}(2, \mathbf{C})$ *bundles $\mathcal{E}$ on X with $c_2(\mathcal{E}) = k$, which can be written in the form of an extension (3.3.1) with $\mathcal{L}$ generic, is parameterised by a complex manifold of dimension $3k$.*

Proof of 3.5.1: Let $S \subset \mathrm{Hilb}^k X$ be the open submanifold corresponding to clusters. Let $\mathcal{Z} \subset S \times X$ denote the universal subscheme so we have a diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & S \\ X & \end{tikzcd}$$

such that $\pi_{2*}(\pi_1^*(s)) \cong Z_s$, where Z_s is the cluster on X given by $s \in S$. Let $\Lambda \subset \mathrm{Pic}(X) \cong \mathbf{C}^*$ denote the open subset of isomorphism classes of bundles on X satisfying (3.3.11) and let $\mathcal{V} \longrightarrow X \times \Lambda$ be the restriction of a Poincaré line bundle. If $\tilde{\mathcal{V}} \longrightarrow \mathcal{Z} \times \Lambda$ is the pull-back of $\mathcal{V}$ by the obvious map, then we have maps

$$\begin{tikzcd} \mathcal{Z} \times \Lambda \arrow[r, "\tilde{\pi}_1"] \arrow[d, "\tilde{\pi}_2"'] & S \times \Lambda \\ X & \end{tikzcd}$$

such that the relative Ext-group,

3.5.2. $$\mathrm{Ext}^1(\pi_1^{-1}(s,\mathcal{L}); \tilde{\mathcal{V}}^{-1}\mathcal{J}_Z, \tilde{\mathcal{V}}) \cong \mathrm{Ext}^1(X; \mathcal{L}^{-1}\mathcal{J}_{Z_s}, \mathcal{L}),$$

has constant dimension $l(Z_s) = k$. By the analogue of Grauert's direct image theorem for Ext-groups [10, Satz 3], there is therefore a rank k vector bundle $\mathcal{U}$ over $S \times \Lambda$ with fibre $\mathcal{U}_{s,\mathcal{L}}$ over a point $(s,\mathcal{L}) \in S \times \Lambda$ given by (3.5.2).

There is a corresponding universal rank 2 torsion-free sheaf $\bar{\mathcal{E}}$ over $\mathbf{P}(\mathcal{U}) \times X$ such that given any $\tilde{\eta} \in \mathcal{U}_{s,\mathcal{L}}$ corresponding to $\eta \in \mathbf{P}(\mathcal{U}_{s,\mathcal{L}})$, we have

3.5.3. $$\bar{\mathcal{E}}|_{\{\eta\}\times X} \cong \mathcal{E},$$

where $\mathcal{E}$ is the torsion free rank 2 sheaf on X defined by the extension class $\tilde{\eta}$.

The space of stable, locally free extensions forms a non-empty open submanifold of $\mathbf{P}(\mathcal{U})$. Thus, the dimension of the space of moduli is given by

$$\begin{aligned} \dim \mathbf{P}(\mathcal{U}) &= k - 1 + \dim(S \times \Lambda) \\ &= k - 1 + (2k+1) \\ &= 3k. \end{aligned}$$

$\boxed{3.5.1}$

3.6. Construction of an exceptional element

Suppose $\mathcal{L}^2 \cong \mathcal{O}_X(-r)$ for some $r \in \mathbf{Z}$ with $r \geq 1$, then it follows from the fundamental exact sequence (3.3.7) that an element η of $H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))$ for which η_x generates $\{\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})\}_x$ as an $\mathcal{O}_{X,x}$-module for each point x in $\mathrm{Supp}Z$, lifts to a global extension if and only if $\delta(\eta) = 0$ in $\mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L})$. The resulting extension is then locally free by Proposition 3.3.8.

Proposition 3.6.1. *Suppose we have a locally free extension $\mathcal{E}$ of the form (3.3.1), where the bundle $\mathcal{L}$ is exceptional, then $\mathcal{L}$ is maximal if and only if $h^0(\mathcal{L}^{-2}\mathcal{J}_Z) \leq 1$.*

Proof of 3.6.1: Suppose $h^0(\mathcal{L}^{-2}\mathcal{J}_Z) \geq 2$; then the linear system $|\mathcal{L}^{-2}\mathcal{J}_Z|$ has a moving part. Since all effective divisors on X are sums of fibres of π, we may choose a fibre F in the moving part such that $F \cap \mathrm{Supp}Z = \emptyset$. From the exact sequence

3.6.2. $$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{L}^{-1}(-F)\mathcal{E} \longrightarrow \mathcal{L}^{-2}(-F)\mathcal{J}_Z \longrightarrow 0,$$

we obtain $H^0(\mathcal{L}^{-1}(-F)\mathcal{E}) \neq 0$, so $\mathcal{L}$ could not have been maximal.

Conversely, if $h^0(\mathcal{L}^{-2}\mathcal{J}_Z) = 0$, then $h^0(\mathcal{L}^{-2}(-m)\mathcal{J}_Z) = 0$ for all $m \geq 0$, and so from the exact sequence in cohomology given by (3.6.2) we see that $\mathcal{E}$ can admit no destabilising bundle of higher degree.

When $h^0(\mathcal{L}^{-2}\mathcal{J}_Z) = 1$, then by Lemma 3.4.7 we have $\deg\mathcal{L}^{-2} = w(Z)$. The exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{L}^{-1}(-1)\mathcal{E} \longrightarrow \mathcal{L}^{-2}(-1)\mathcal{J}_Z \longrightarrow 0$$

then gives $H^0(\mathcal{L}^{-1}(-1)\mathcal{E}) = H^0(\mathcal{L}^{-2}(-1)\mathcal{J}_Z)$, but the last term is 0 by Lemma 3.4.7, and so $\mathcal{L}$ must be maximal. $\boxed{3.6.1}$

Corollary 3.6.3. *If a stable rank 2 bundle $\mathcal{E}$ can be expressed as an extension of the form (3.3.1), where $\mathcal{L}$ is its maximal destabilising bundle and $\mathcal{L}$ is exceptional, then $\mathcal{L}$ must satisfy*

$$\deg \mathcal{L} < 0,$$

and

$$\deg \mathcal{L}^{-2} + 1 - w(Z) \leq 1;$$

i.e.,

3.6.4.
$$-\frac{w(Z)}{2} \leq \deg \mathcal{L} < 0.$$

Conversely, given any exceptional bundle satisfying (3.6.4), any locally free extension of the form (3.3.1) is stable.

Note that from the proof of (3.6.1), it follows that if $\mathcal{L}^2 \cong \mathcal{O}_X(-r)$ for some $r \in \mathbb{Z}$ and $\mathcal{E}$ is a locally free extension of the form (3.3.1), for some cluster Z on X, with $\mathcal{E}$ stable, then

3.6.5.
$$H^0(\mathcal{L}^{-2}\mathcal{K}_X\mathcal{J}_Z) = 0.$$

We estimate the number of stable bundles $\mathcal{E}$ of this form with $c_2(\mathcal{E}) = k > 0$ as follows. In constructing such an $\mathcal{E}$, one first has to choose an exceptional line bundle $\mathcal{L}$ with $\mathcal{L}^2 \cong \mathcal{O}_X(-r)$, where to ensure stability $r \in \mathbb{Z}$ must satisfy

$$-\frac{k}{2} \leq r < 0.$$

There is only a discrete set of such bundles. Next, again to ensure stability, one has to choose a cluster Z on X with $l(Z) = k$ from amongst those which satisfy

$$h^0(\mathcal{L}^{-2}\mathcal{J}_Z) \leq 1,$$

for the given $\mathcal{L}$. The number of moduli for Z is clearly bounded above by $2k$. Once such a Z has been fixed, we then have to select a class in $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ which gives rise to a locally free extension. Taking the alternating sum of the dimensions of the spaces appearing in the exact sequence (3.3.5) and using (3.6.5) gives

$$\begin{aligned}\dim \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) &= h^1(\mathcal{L}^2) - h^2(\mathcal{L}^2) + l(Z) \\ &= \chi(\mathcal{L}^2) + l(Z).\end{aligned}$$

The first term vanishes by Riemann-Roch, hence the number of moduli of locally free extensions for a fixed pair $(\mathcal{L}, Z)$ is bounded above by $l(Z) = k$. The isomorphism class of any such extension is then uniquely determined by the class of the extension modulo the action of Aut$\mathcal{L} \cong \mathcal{O}_X^*$. Adding together the various contributions and subtracting 1 for the action of Aut$\mathcal{L}$ shows that the number of moduli is bounded above by $3k - 1$. Comparing this with the value of $3k$ obtained in the "generic case" justifies the classification (3.3.11) and (3.3.12).

3.7. Charge 1 instantons: the lower stratum

Elements of $\mathcal{M}_1^0$ consist of isomorphism classes of stable $\mathrm{SL}(2,\mathbf{C})$ bundles $\mathcal{E}$ with $c_2(\mathcal{E}) = 1$ which can be expressed in the form of a sheaf extension (3.3.1). By Lemma 3.1.7, we have $l(Z) = c_2(\mathcal{E}) = 1$, so Z consists simply of a point $x_0 \in X$, and $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}_Z = \mathbf{C}_{x_0}$, where $\mathbf{C}_{x_0}$ denotes the skyscraper sheaf supported at x with fibre $\mathbf{C}$. Lemma 3.2.2 shows that $\Sigma_1 = \emptyset$ and so the point x_0 is uniquely determined by $\mathcal{L}$. The fibre weight of $\pi^{-1}(y)$ is then zero for all $y \in \mathbf{P}^1$, except for $y_0 = \pi(x_0)$, when the fibre weight of $\pi^{-1}(y_0)$ is equal to 1. Thus $w(Z) = 1$. By (3.6.4) and (3.4.13) the degree of $\mathcal{L}$ must satisfy

3.7.1. $$-1 < \deg \mathcal{L} < 0,$$

and $H^1(\mathcal{L}^2) = H^2(\mathcal{L}^2) = 0$. By (3.6.3) and (3.4.11), $\mathcal{L}$ is a maximal destabilising line bundle of $\mathcal{E}$.

Conversely, given any point $x_0 \in X$, any cluster Z on X supported at x_0 with $l(Z) = 1$ is uniquely determined; namely $\mathcal{J}_Z \cong \mathcal{J}_{x_0}$. Given any line bundle $\mathcal{L}$ on X satisfying (3.7.1), by Proposition 3.3.8, a rank 2 sheaf $\mathcal{E}$ constructed as an extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0,$$

classified by an element

$$\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \cong H^0(\mathcal{L}^2\mathcal{O}_Z) \cong \mathcal{L}^2_{x_0},$$

is locally free if and only if $\eta \in \mathcal{L}^2_{x_0}$ generates $\mathcal{L}^2_{x_0}$ as an $\mathcal{O}_{x_0} = \mathbf{C}_{x_0}$ module; i.e., if and only if $\eta \neq 0$. The resulting $\mathrm{SL}(2,\mathbf{C})$ bundle $\mathcal{E}$ is then stable if and only if $0 > \deg \mathcal{L} > -1$.

Since $\mathrm{Aut}(\mathcal{L}) \cong \mathbf{C}^*$ acts transitively on $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \setminus \{0\} \cong (\mathcal{L}^2_{x_0})^*$, the isomorphism class of the resulting bundle $\mathcal{E}$ is independent of the choice of (non-zero) extension.

Hence we have proved:

Proposition 3.7.2. *If $\mathcal{E}$ is a stable* $\mathrm{SL}(2,\mathbf{C})$ *bundle representing a class in $\mathcal{M}_1^0$, then $\mathcal{E}$ may be constructed as a sheaf extension*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_{x_0}, \longrightarrow 0$$

where $x_0 \in X$ is a point and $\mathcal{L}$ is a line bundle on X satisfying $0 > \deg \mathcal{L} > -1$. *The extension is defined by a non-zero element of* $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_{x_0}, \mathcal{L})$. *The isomorphism class of $\mathcal{E}$ as an* $\mathrm{SL}(2,\mathbf{C})$ *bundle is independent of this choice of extension.*

We now investigate to what extent the isomorphism class of a bundle $\mathcal{E}$ constructed as above depends on the choice of the data $(\mathcal{L}, x_0)$. Clearly, the isomorphism class of $\mathcal{E}$ depends only on the choice of $\mathcal{L}$ up to isomorphism, so the construction defines a map

$$\Theta : \mathbf{A} \times X \longrightarrow \mathcal{M}_1^0,$$

where $\mathbf{A} \subset \mathrm{Pic}(X) = \mathbf{C}^*$ is the annulus given by

$$\mathbf{A} = \{\mathcal{L} \in \mathrm{Pic}(X) : -1 < \deg \mathcal{L} < 0\} = \{\mu \in \mathbf{C}^* : \lambda^{-1} < |\mu| < 1\}.$$

Suppose $\Theta(\mathcal{L}_1, x_1) = \Theta(\mathcal{L}_2, x_2) = \mathcal{E}$, so $\mathcal{E}$ can be written as two (generally different) extensions,

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_1^{-1}\mathcal{J}_{x_1} \longrightarrow 0;$$

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2^{-1}\mathcal{J}_{x_2} \longrightarrow 0,$$

where $\mathcal{L}_1, \mathcal{L}_2$ are maximal destabilising line bundles of $\mathcal{E}$. Hence, by Proposition (3.4.11), we have $\mathcal{L}_1 \cong \mathcal{L}_2$ or $\mathcal{L}_1 \cong \mathcal{L}_2^{-1}(-1)$. The points x_i are the uniquely defined zero-sets of any non-trivial section of the bundles $\mathcal{L}_i^{-1}\mathcal{E}$. As in Section 3.2, $\mathcal{E}$ restricts to a bundle of type (3) on a fibre $f = \pi^{-1}(y)$ for some $y \in \mathbf{P}^1$, where $y = \pi(x_1) = \pi(x_2)$. If $\mathcal{E}|_{\pi^{-1}(y)} \cong \mathcal{V} \oplus \mathcal{V}^{-1}$ where $\mathcal{V} \in \mathrm{Pic}^1(f)$, then x_1 and x_2 are respectively the unique points on $\pi^{-1}(y)$ where any non-trivial section of $\mathcal{L}_1^{-1}|_f \otimes \mathcal{V}$ or $\mathcal{L}_2^{-1}|_f \otimes \mathcal{V}$ vanishes. Thus we have

$$\mathcal{L}_1^{-1}|_f \otimes \mathcal{V} \cong \mathcal{O}_f(x_1) \quad \text{and} \quad \mathcal{L}_2^{-1}|_f \otimes \mathcal{V} \cong \mathcal{O}_f(x_2).$$

If $\mathcal{L}_1 \cong \mathcal{L}_2$, then we must have $x_1 = x_2$ by uniqueness. If $\mathcal{L}_1 \cong \mathcal{L}_2^{-1}(-1)$, then $\mathcal{L}_1|_f \cong \mathcal{L}_2^{-1}|_f$, and so, if $\mathcal{L}_1|_f \cong \mathcal{O}_f(P)$ for some degree 0 divisor P on f, we have

$$\mathcal{O}_f(x_1 + P) \cong \mathcal{O}_f(x_2 - P),$$

and consequently [32, p.297]

3.7.3. $$x_1 = x_2 - 2P.$$

Using the notation of Subsection 2.5.1, up to isomorphism we have $\mathcal{L}_1 = \mathcal{L}_\mu$ and $\mathcal{L}_2 = \mathcal{L}_\nu$ for some $\mu, \nu \in \mathbf{C}^*$, with $\lambda^{-1} < |\mu|, |\nu| < 1$. In the case $\mathcal{L}_1 \cong \mathcal{L}_2^{-1}(-1)$ this gives $\mu = \nu^{-1}\lambda^{-1}$. Thus $\mathcal{L}_1|_f \cong \mathcal{V}_\mu$ and $\mathcal{L}_2|_f \cong \mathcal{V}_\nu$ and the equation (3.7.3) becomes

$$x_2 = x_1.\langle \mu^2 \rangle,$$

where $\langle \mu^2 \rangle$ denotes the image of μ^2 in $\mathrm{Pic}^0(T) \cong T$ under the standard identification, and we use the principal T-structure of X.

Thus $\Theta(\mathcal{L}_1, x_1) = \Theta(\mathcal{L}_2, x_2)$ if and only if

3.7.4. $$\{\mathcal{L}_1 \cong \mathcal{L}_2 \text{ and } x_1 = x_2\} \text{ or } \{\mathcal{L}_1 \cong \mathcal{L}_2^{-1}(-1) \text{ and } x_2 = x_1.\langle \mathcal{L}_1^2 \rangle\}.$$

The identity component of the automorphism group of X, $\mathrm{Aut}_0(X)$, acts naturally on the left of $\mathcal{M}_k$. The action on X is by morphisms of elliptic fibrations and so divisors of $\mathrm{SL}(2,\mathbf{C})$ bundles, which are divisors on $\mathbf{P}^1 \times \mathbf{P}^1$, are simply "translated" by automorphisms of the first $\mathbf{P}^1$-factor. Hence the stratification of $\mathcal{M}_k$ by graph type is preserved. Restricting to the action of $X \subset \mathrm{Aut}_0(X)$, this leads to:

Theorem 3.7.5. *The natural action of $X \subset \mathrm{Aut}_0(X)$ on $\mathcal{M}_1^0$ is free, and the space $\mathcal{M}_1^0$ is diffeomorphic as a principal X-bundle to $X \times (\mathbf{P}^1 \setminus I)$, where $I \subset \mathbf{P}^1$ is a closed subset of $\mathbf{P}^1$ homeomorphic to the interval* $[0,1]$.

Proof of 3.7.5: Recall that $\mathbf{P}^1$ is the image of $\mathbf{C}^*$ under the successive quotients

$$\mathbf{C}^* \longrightarrow T \xrightarrow{q} \mathbf{P}^1.$$

The map q is a $2:1$ cover, ramified at the half-periods $\pm 1, \pm\sqrt{\lambda}$ in $T = \mathbf{C}^*/\mathbf{Z}$. The circles $S_n = \{z \in \mathbf{C}^* : |z| = \lambda^n\}$, for $n \in \mathbf{Z}$, are mapped onto a circle in T passing through the two half-periods ± 1. Under q this circle is mapped to the closed set I homeomorphic to a closed interval with endpoints $q(1)$ and $q(-1)$. Under the the composition of the inclusion $\mathbf{A} \hookrightarrow \mathbf{C}^*$ and q, the annulus $\mathbf{A}$ maps onto $\mathbf{P}^1 \setminus I$ and the condition that two points, $\mu, \nu \in \mathbf{A}$ have the same image in $\mathbf{P}^1$ is precisely that $\mu = \nu$, or $\mu = \nu^{-1}\lambda^{-1}$; i.e., that μ and ν are equivalent under the $\mathbf{Z}_2$-action on $\mathbf{A}$ generated by $\mu \mapsto \mu^{-1}\lambda^{-1}$.

By (3.7.4), the holomorphic map $\Theta : \mathbf{A} \times X \longrightarrow \mathcal{M}_1^0$ is invariant under the $\mathbf{Z}_2$-action generated by

$$(\mu, x) \mapsto (\mu^{-1}\lambda^{-1}, x\mu^2),$$

which lifts the action of $\mathbf{Z}_2$ on $\mathbf{A}$. The group $\mathbf{Z}_2$ acts trivially on the fibres of the projection $\mathbf{A} \times X \to \mathbf{A}$ over the fixed points $\mu = \pm\sqrt{\lambda}$ of its action on $\mathbf{A}$. Hence, the quotient $\mathbf{B} = (\mathbf{A} \times X)/\mathbf{Z}_2$ is a fibre bundle over $\mathbf{A}/\mathbf{Z}_2 = \mathbf{P}^1 \setminus I$ with fibre X. We then have a one-to one, surjective holomorphic map

$$\overline{\Theta} : \mathbf{B} \longrightarrow \mathcal{M}_1^0,$$

which by [3, Theorem 5.13] is a diffeomorphism.

The Lie group X acts holomorphically on the left of itself, and so on the left of $\mathcal{M}_1^0$ by $\mu.\mathcal{E} = (\mu^{-1})^*\mathcal{E}$. If $\mathcal{E} \in \mathcal{M}_1^0$ is given as an extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_x \longrightarrow 0,$$

and if $\mu \in X$, then $(\mu^{-1})^*\mathcal{E}$ is given as an extension

$$0 \longrightarrow (\mu^{-1})^*\mathcal{L} \longrightarrow (\mu^{-1})^*\mathcal{E} \longrightarrow (\mu^{-1})^*\mathcal{L}^{-1}\mathcal{J}_{\mu x} \longrightarrow 0.$$

Since the action of X on $\mathrm{Pic}(X)$ is trivial, this is just

$$0 \longrightarrow \mathcal{L} \longrightarrow (\mu^{-1})^*\mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_{\mu x} \longrightarrow 0.$$

Hence, if we endow $\mathbf{A} \times X$ with X-action given by

$$\mu \cdot (\mathcal{L}, x) = (\mathcal{L}, \mu . x),$$

the map Θ is X-equivariant. The left action of X commutes with the right action of $\mathbf{Z}_2$, and descends to $\mathbf{B}$ giving it the structure of a principal X-fibration:

$$\mathbf{B} \longrightarrow \mathbf{P}^1 \setminus I.$$

Thus, we have a diffeomorphism

$$\overline{\Theta} : \mathbf{B} \longrightarrow \mathcal{M}_1^0,$$

preserving the smooth fibre bundle structure. The result follows from the contractibility of $\mathbf{P}^1 \setminus I$. $\boxed{3.7.5}$

Chapter 4

Monopoles on $D^2 \times S^1$

4.1. Description as a sheaf extension

In their paper [16], Braam and Hurtubise investigated the moduli space of magnetic SU(2) monopoles on $D^2 \times S^1$, where D^2 is the open 2-disc with its standard hyperbolic metric, and S^1 has its usual metric. As explained in [14, 15, 16], these monopoles may be identified with $\tilde{S}^1$-invariant instantons on $S^1 \times S^3$, or equivalently with $\tilde{\mathbf{C}}^*$-equivariant stable $\mathrm{SL}(2, \mathbf{C})$ bundles on the Hopf surface X. Here $\tilde{S}^1, \tilde{\mathbf{C}}^*$ denote the double covers of $S^1, \mathbf{C}^*$ respectively. We think of $\tilde{\mathbf{C}}^*$ as a copy of $\mathbf{C}^*$ with the double cover $\mathbf{C}^* \to \mathbf{C}^*$ being given by $\omega \mapsto \omega^2$. The action of $\omega \in \tilde{\mathbf{C}}^*$ on $(z_0, z_1) \in X$ is defined by

$$\omega.(z_0, z_1) = (\omega^2 z_0, z_1).$$

This induces a corresponding $\tilde{\mathbf{C}}^*$-action on $\mathbf{P}^1$ with respect to which the projection $\pi : X \to \mathbf{P}^1$ is equivariant. The points $0 \stackrel{\text{def}}{=} [0, 1]$ and $\infty \stackrel{\text{def}}{=} [1, 0]$ of $\mathbf{P}^1$ are fixed by the action. The stabilisers of the remaining points are all $\{\pm 1\}$. It is easy to see that $\tilde{\mathbf{C}}^*$ leaves every point of the fibre $T_0 = \pi^{-1}(0)$ fixed. On the fibre $T_\infty = \pi^{-1}(\infty)$ each point has stabiliser $\mathbf{Z} \times \mathbf{Z}_2 \cong \{\pm \lambda^{j/2} : j \in \mathbf{Z}\}$.

We think of a $\tilde{\mathbf{C}}^*$-equivariant bundle $\mathcal{V}$ over X as a bundle carrying a $\tilde{\mathbf{C}}^*$-action covering that on X. Alternatively, letting

$$A : \tilde{\mathbf{C}}^* \times X \to X$$

denote the action and

$$\pi_2 : \tilde{\mathbf{C}}^* \times X \to X$$

the projection, we may think of the action on $\mathcal{V}$ as an isomorphism

$$\alpha : \mathcal{V} \to A^* \mathcal{V}$$

which satisfies the following. For each $g \in \tilde{\mathbf{C}}^*$, restricting α to $\{g\} \times X$ gives an isomorphism

$$\alpha_g : \mathcal{V} \to g^* \mathcal{V}.$$

By a slight abuse of notation, we require

4.1.1. $$\alpha_g \circ \alpha_h = \alpha_{gh}$$

for each $g, h \in \mathbf{C}^*$. (Here α_g really refers to the natural isomorphism $h^*\mathcal{V} \to (gh)^*\mathcal{V} = h^*g^*\mathcal{V}$ induced by α_g.)

In [16] it is shown that any $\mathbf{C}^*$-equivariant stable $\mathrm{SL}(2,\mathbf{C})$ bundle $\mathcal{E}$ on X defines an element of $\mathcal{M}_k^0$, where $k = c_2(\mathcal{E})$ and the divisor of $\mathcal{E}$ is of the form

$$D(\mathcal{E}) = \mathbf{P}^1 \times \{l\} + kF_0,$$

where $l \in \mathbf{P}^1$ and $F_0 = [0,1] \times \mathbf{P}^1$. This is essentially due to the fact that since $\mathbf{C}^*$ leaves every point in the elliptic curve T_0 fixed, any $\mathbf{C}^*$-equivariant bundle then splits over T_0 as a bundle of type (3):

4.1.2. $$\mathcal{E}|_{T_0} \cong \mathcal{L}_n \oplus \mathcal{L}_n^{-1},$$

for some $\mathcal{L}_n \in \mathrm{Pic}^n(T_0)$, where $1 \leq n \leq k$. The group $\mathbf{C}^*$ acts as automorphisms of $\mathcal{E}$ over T_0, preserving the splitting (4.1.2). If $\mathbf{C}^*$ acts on $\mathcal{L}_n$ with weight $-m \in \mathbf{Z}$, then $k = mn$. The integers m, n are the *mass* and *charge* of the monopole respectively.

Let $\mathcal{M}(m,n) \subset \mathcal{M}_{mn}^0$ denote the moduli space of monopoles of charge n and mass m; i.e., the space of isomorphism classes of stable $\mathbf{C}^*$-equivariant $\mathrm{SL}(2,\mathbf{C})$ bundles over X. Any bundle $\mathcal{E}$ representing a class in $\mathcal{M}(m,n)$ fits into an exact sequence of sheaves

4.1.3. $$0 \longrightarrow \mathcal{L} \xrightarrow{\otimes s} \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0,$$

where $\mathcal{L}$ is a maximal destabilising line bundle of $\mathcal{E}$ and $(Z, \mathcal{O}_Z)$ is the zero-scheme of a non-trivial section s of $\mathcal{L}^{-1}\mathcal{E}$. It is shown in [16] that we may give $\mathcal{L}$ a $\mathbf{C}^*$-structure such that the section s is $\mathbf{C}^*$-equivariant. A $\mathbf{C}^*$-structure on a given line bundle $\mathcal{L}$ is uniquely determined by two items of data, namely the weight r of the action on the fibres of $\mathcal{L}$ over T_0 and the monodromy μ for the action around the loop in T_∞ defined by the path

$$\begin{aligned} \gamma : [0,1] &\to T_\infty, \\ \gamma(t) &= t\lambda^{1/2}. \end{aligned}$$

If $\mathcal{L}$ corresponds to the element $\nu \in \mathrm{Pic}(X) \cong \mathbf{C}^*$, then we have

4.1.4. $$\nu = \lambda^{r/2}\mu^{-1}.$$

Restricting (4.1.3) to T_0, we see $\mathcal{L}_n \cong \mathcal{L} \otimes \mathcal{O}_{T_0}(Z_0)$, where Z_0 is the zero-divisor of degree n of the section s restricted to T_0. Hence the line bundle $\mathcal{L}$ must have weight $-m$. The monodromy μ is then determined by (4.1.4).

Associated to the isomorphism class of the bundle $\mathcal{E}$ we therefore have three items of data:

(i) A cluster $(Z, \mathcal{O}_Z)$ on X with $\mathrm{Supp} Z \subset T_0$ and $\deg(Z_0) = n$.

(ii) A $\tilde{\mathbf{C}}^*$-equivariant line bundle $\mathcal{L}$ on X with weight $-m$.

(iii) An extension class $\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$.

Of course $\tilde{\mathbf{C}}^*$-equivariance imposes further conditions on Z and η, but for general values of n the structure of Z will in any case be quite complicated. However, in the case when the monopole charge $n = 1$, the structure is sufficiently simple to permit an explicit description of the moduli space. Thus, from now on we restrict our attention to the moduli space $\mathcal{M}(m, 1)$ of charge 1 monopoles with mass m.

Lemma 4.1.5. *Suppose $\mathcal{E}$ in (4.1.3) represents a class in $\mathcal{M}(m, 1)$, then $H^0(\mathcal{L}^{-1}\mathcal{E}) = 1$ and the zero scheme of any non-trivial section s of $\mathcal{L}^{-1}\mathcal{E}$ is supported at a single, uniquely-determined point $x_0 \in T_0$.*

Proof of 4.1.5: The statement $H^0(\mathcal{L}^{-1}\mathcal{E}) = 1$ follows from (3.2.2). Since $s|_{T_0}$ vanishes only at points in the support of Z and $\mathcal{E}|_{T_0}$ splits as $\mathcal{E}|_{T_0} \cong \mathcal{L}_1 \oplus \mathcal{L}_1^{-1}$ with $\deg \mathcal{L}_1 = 1$, the section s must vanish at exactly one point on T_0. $\boxed{4.1.5}$

Let $\mathcal{E}$ be as in (4.1.5) and let $x_0 \in T_0$ be the corresponding point. The constraints (i) and (ii) on Z and $\tilde{\mathbf{C}}^*$-equivariance allow us to determine Z completely.

Let $z = x_0/x_1$ denote the standard affine coordinate on

$$U_0 = \{[x_0, x_1] \in \mathbf{P}^1 : x_1 \neq 0\},$$

then $\pi^{-1}(U_0) \cong \mathbf{C} \times T$ via the map

$$(x_0, x_1) \mapsto (x_0/x_1, (x_1)).$$

The induced $\tilde{\mathbf{C}}^*$-action on $\mathbf{C} \times T$ is then given by

$$\omega.(z, t) = (\omega^2 z, t).$$

The following result is an easy extension of [5, Propositions 6.1 and 6.2]:

Lemma 4.1.6. *Suppose $\mathcal{V} \to \mathbf{C} \times \mathbf{C}$ is a $\mathbf{C}^*$-equivariant holomorphic vector bundle, where $\mathbf{C} \times \mathbf{C}$ has the $\mathbf{C}^*$-action given by*

$$\omega.(z, \zeta) = (\omega z, \zeta);$$

then $\mathcal{V}$ is equivariantly isomorphic to $\mathbf{C} \times \mathbf{C} \times V_0$, where V_0 is the $\mathbf{C}^*$ representation given by the fibre of $\mathcal{V}$ over $(0, 0)$.

Suppose $x_0 = (0, t_0) \in T_0$. Choosing an open disc U in T with local coordinate t centred on t_0, then by (4.1.6) the bundle $\mathcal{L}^{-1}\mathcal{E}|(\mathbf{C} \times U)$ is $\tilde{\mathbf{C}}^*$-isomorphic to $\mathbf{C} \times U \times (V_1 \oplus V_2)$, where V_1, V_2 are representations of $\tilde{\mathbf{C}}^*$ of weights 0 and $2m$ respectively. A choice of basis vectors for V_1 and V_2 determines a $\tilde{\mathbf{C}}^*$-equivariant trivialisation for $\mathcal{L}^{-1}\mathcal{E}$ over $\mathbf{C} \times U$. The components (f, g) of

s in V_1 and V_2 respectively then define a pair of $\mathbf{C}^*$-equivariant holomorphic functions on $\mathbf{C} \times U$ such that

$$\{x_0\} = \{x \in U \times \mathbf{C} : f(x) = g(x) = 0\}$$

and h.c.f.$(f, g) = 1$.

We use this to prove the following:

Lemma 4.1.7. *If $\mathcal{E} \in \mathcal{M}(m,1)$ is given as an extension as in (4.1.3), then with respect to the local coordinates (z,t) defined above, the ideal sheaf $\mathcal{J}_Z$ is generated near x_0 by $\{t, z^m\}$.*

Proof of 4.1.7: By $\mathbf{C}^*$-equivariance,

$$f(\omega^2 z, t) = f(z,t),$$

for all $\omega \in \mathbf{C}^*$, so $f(z,t) = F(t)$ for some holomorphic function F depending on t only. Similarly, we obtain $g(z,t) = z^m G(t)$ for some holomorphic function G depending only on t. The assumption on the order of vanishing of s along T_0 gives $F(t) = tu(t)$ for some holomorphic function u with $u(0) \neq 0$. By the assumption on common factors, we must have $G(0) \neq 0$, and so we see that the functions t and z^m generate $\mathcal{J}_Z$ near x_0. $\boxed{4.1.7}$

We now examine to what extent the extension class $\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ is determined by the group action. From (4.1.3) we obtain a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & g^*\mathcal{L} & \longrightarrow & g^*\mathcal{E} & \longrightarrow & (g^*\mathcal{L})^{-1}\mathcal{J}_Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_g^{-1}} & & \downarrow{\scriptstyle \alpha_g^{-1}} & & \downarrow{\scriptstyle \alpha_g^{-1}} & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}^{-1}\mathcal{J}_Z & \longrightarrow & 0
\end{array}$$

where the bottom row is defined by the class $\alpha_g^{-1}(g^*\eta)$. We must have $\alpha_g^{-1}(g^*\eta) = a.\eta$, for some scalar $a \in \mathbf{C}^*$ (i.e., the extension class is projectively invariant). The transformations $\alpha_g^{-1} \circ g^*$ define an action of $\mathbf{C}^*$ on the terms of the exact sequence

4.1.8.
$$0 \to H^1(\mathcal{L}^2) \to \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \xrightarrow{r} H^0(\underline{\mathcal{E}xt}^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})) \to H^2(\mathcal{L}^2) \to \dots$$

making it into an exact sequence of $\mathbf{C}^*$-modules.

Let $V = \mathbf{C} \times U$ be the neighbourhood of x_0 as in (4.1.7); then we have a resolution for $\mathcal{L}^{-1}\mathcal{J}_Z$ over V:

4.1.9. $$0 \to \mathcal{L}_V \xrightarrow{A} \mathcal{L}_V \oplus \mathcal{L}_V^{-1} \xrightarrow{B} \mathcal{L}_V^{-1} \otimes \mathcal{J}_Z \to 0.$$

Using Lemma 4.1.6, we may take the sheaf $\mathcal{L}_V$ to be $\mathcal{O}_V \otimes L$, where L is a 1-dimensional $\mathbf{C}^*$-module of weight $-m$, in which case A and B can be identified with matrices of functions holomorphic on V:

$$A_{(z,t)} = \begin{pmatrix} t & z^m \end{pmatrix} \quad B_{(z,t)} = \begin{pmatrix} z^m \\ -t \end{pmatrix}$$

respectively. The sequence (4.1.9) is then $\mathbb{C}^*$-equivariant. By definition $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ is given by

4.1.10. $$\mathcal{H}om(\mathcal{L}, \mathcal{L})/(A^*\mathcal{H}om(\mathcal{L}^{-1} \oplus \mathcal{L}, \mathcal{L})).$$

Now $\mathcal{H}om(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_V \otimes \mathrm{Hom}(L, L)$ is the sheaf of sections of a weight 0 bundle over V. If $(\alpha, \beta) \in \mathcal{H}om(\mathcal{L}^{-1} \oplus \mathcal{L}, \mathcal{L})$, we have

$$A^*(\alpha, \beta)f = (\alpha, \beta)Af = (t\alpha + z^m\beta)f.$$

Hence, using (4.1.10)

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \cong \mathcal{O}_V/\{t, z^m\} \cong \{1, z, \ldots, z^{m-1}\} = \mathcal{O}_Z$$

decomposes under the action of $\mathbb{C}^*$ into 1-dimensional modules of weights $2r$ spanned by z^r for $r = 0, \ldots, m-1$. This leads to

Lemma 4.1.11. *If $\mathcal{E}$ is a $\mathbb{C}^*$-equivariant locally free extension of the form (4.1.8) determined by an element $\eta \in \mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$, then the localisation $\rho(\eta) \in H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))$ is invariant under the action of $\mathbb{C}^*$ and is a scalar multiple of the canonical element $1_Z \in \mathcal{O}_Z$ given by the isomorphism $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \cong \mathcal{O}_Z$.*

Proof of 4.1.11: We know by Proposition 3.3.8 that $\rho(\eta)$ must be a unit in $\mathcal{O}_Z$ and by the above discussion, it must be projectively invariant under the action of $\mathbb{C}^*$. It must therefore be a weight vector. From the above, we see the only such vectors are scalar multiples of 1_Z. $\boxed{4.1.11}$

Hence associated to each monopole bundle $\mathcal{E} \in \mathcal{M}(m, 1)$ we can canonically associate the following data:

(a) A cluster Z supported at a point $x_0 \in T_0$ whose ideal sheaf $\mathcal{J}_Z$ has local generators $\{t, z^m\}$.

(b) A $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ of weight $-m$ on X.

(c) A local extension class $\nu.1_Z \in \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}) \cong \mathcal{O}_Z$, where $\nu \in \mathbb{C}^*$.

Note that the parameter $\nu \in \mathbb{C}^*$ is ineffective: rescaling results in an isomorphic bundle.

4.2. Charge 1 monopoles

Given a cluster $(Z, \mathcal{O}_Z)$ on X supported at $x_0 \in T_0$, whose ideal sheaf $\mathcal{J}_Z$ has generators as in (4.1.7), and a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on X, then we wish to construct a locally free $\mathbb{C}^*$-equivariant extension of the form:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1}\mathcal{J}_Z \longrightarrow 0.$$

As we have seen, such extensions are defined by a class η in $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$ whose localisation

$$\rho(\eta) \in \mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L}) = H^0(\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L})) = \{\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L})\}_{x_0}$$

is a non-zero scalar multiple of the canonical generator 1_Z of the $\mathcal{O}_{X,x_0}$-module $\{\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L})\}_{x_0}$ under the isomorphism

$$\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L}) \cong (\mathcal{O}_Z)_{x_0}.$$

Grothendieck-Serre duality in (3.3.7) is given by the non-degenerate global residue pairing [30, p.707 - 708]:

4.2.1. $$\mathrm{Res} : H^0(\mathcal{O}_Z\mathcal{L}^{-2}\mathcal{K}_X) \otimes \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{L}^2) \longrightarrow \mathbf{C}.$$

The generators $(t, z^m) \in \mathcal{O}_{X,x_0}$ defining $\mathcal{J}_Z$ and a local trivialisation for $\mathcal{L}^{-2}\mathcal{K}_X$ gives an isomorphism

$$\mathcal{O}_Z\mathcal{L}^{-2}\mathcal{K}_X \cong \mathcal{O}_Z,$$

such that the pairing (4.2.1) is given in local coordinates by the local residue pairing at 0 [30, p.690, p.693]

4.2.2. $$\mathrm{Res}_0 : \mathcal{O}_Z \otimes \mathcal{O}_Z \longrightarrow \mathbf{C}.$$

The generator 1_Z of $\mathcal{O}_Z$ corresponds via this pairing to the map

4.2.3. $$\mathrm{Res}_0(\cdot, 1_Z) : \mathcal{O}_Z \longrightarrow \mathbf{C},$$

in $H^0(\mathcal{O}_Z\mathcal{L}^{-2}\mathcal{K}_X)^*$. In order that this generator lift to a global extension, we require that $\delta(1_Z) = 0$ in $\mathrm{Ext}^2(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L})$ or equivalently

4.2.4. $$\mathrm{Res}_0(q(\sigma), 1_Z) = 0,$$

for all global sections σ of $\mathcal{L}^{-2}\mathcal{K}_X$.

In the case where $\mathcal{L}^2 \not\cong \mathcal{O}_X(r)$ for any $r \in \mathbf{Z}$, there are no global sections of $\mathcal{L}^{-2}\mathcal{K}_X$, and so locally free extensions always exist. If, however, the bundle $\mathcal{L}^{-2}$ is isomorphic to $\mathcal{O}_X(r)$ for some $r \in \mathbf{Z}$, then $\mathcal{L}^{-2}\mathcal{K}_X \cong \mathcal{O}_X(r-2)$. Global sections of this bundle are given locally with respect to the trivialisation on $U_0 \times T$ by functions of the form

4.2.5. $$a_0 + a_1 z + \ldots + a_{r-2}z^{r-2},$$

where $a_0, \ldots, a_{r-2} \in \mathbf{C}$ are constants. Working in the local coordinates (z, t) defined earlier, since $\mathcal{J}_Z$ is generated by $\{t, z^m\}$, the local residue pairing (4.2.2) is given by

4.2.6. $$\mathrm{Res}_0(f, g) = \frac{1}{4\pi^2}\int_\gamma \frac{\tilde{f}(z,t)\tilde{g}(z,t)}{tz^m}dzdt,$$

where $\tilde{f}, \tilde{g}$ are lifts of $f, g \in \mathcal{O}_Z$ to holomorphic functions on some neighbourhood of x_0 containing the polycylinder γ defined by $|t| = \epsilon_1, |z| = \epsilon_2$, for $\epsilon_1, \epsilon_2 > 0$ chosen suitably small [30, p.657]. The residue pairing (4.2.3) applied to a polynomial of the form (4.2.5) picks out the coefficient of z^{m-1}. Hence condition (4.2.4) is satisfied if and only if $m-1 > r-2$; i.e., $m > r-1$. Thus, in the exceptional case, a locally free extension exists if and only if

$$\deg \mathcal{L} > -(m+1)/2$$

or equivalently,

$$\deg \mathcal{L} \geq -m/2.$$

Since $\mathcal{J}_Z$ is generated by the functions t, z^m, we have:

$$w(Z) = w_{T_0}(Z) = m.$$

By (3.4.13), in the generic case, the bundle $\mathcal{E}$ constructed will be stable if and only if

$$-m < \deg \mathcal{L} < 0.$$

In the exceptional case, where $\mathcal{L}^2$ is isomorphic to $\mathcal{O}_X(-r)$ for some $r \in \mathbf{Z}$ with $0 < r \leq m$, we have $H^0(\mathcal{L}^{-2}\mathcal{J}_Z) = 0$ unless $r = m$, when $H^0(\mathcal{L}^{-2}\mathcal{J}_Z) = 1$. Hence by (3.6.1), the bundle $\mathcal{E}$ is stable.

Hence we have shown:

Lemma 4.2.7. *Suppose Z is the cluster on X supported at a point $x_0 \in T_0$ and whose ideal sheaf in the coordinates defined above has local generators $\{t, z^m\}$. Let $\mathcal{L}$ be a line bundle on X and suppose $\deg \mathcal{L} > -(m+1)/2$ if $\mathcal{L}$ is exceptional. If η in $H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))$ is a non-zero scalar multiple of the canonical generator of $\mathcal{O}_Z$ under the isomorphism*

$$\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{O}_Z, \mathcal{L}) \cong \mathcal{O}_Z,$$

then η lifts to a global extension class $\tilde{\eta}$ in $\operatorname{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$.

The lift $\tilde{\eta}$ of Lemma 4.2.7 is uniquely defined when $\mathcal{L}$ is generic. If $\mathcal{L}$ is exceptional, then $\tilde{\eta}$ is defined only up to an element of $H^1(\mathcal{L}^2)$. Since the element η is invariant under the $\mathbf{C}^*$-action, using the exact sequence (4.1.8) we see that $\tilde{\eta}$ is of the form $\eta_0 + \beta$ where η_0 is $\mathbf{C}^*$-invariant and $\beta \in H^1(\mathcal{L}^2)$. Since $\mathbf{C}^*$-equivariant extensions are projectively $\mathbf{C}^*$-invariant, we see that there will be a unique lift which leads to an equivariant extension provided $H^1(\mathcal{L}^2)$ contains no non-trivial elements invariant under the action of $\mathbf{C}^*$.

Lemma 4.2.8. *If $k > 0$ and $\mathcal{O}_X(-k)$ is given the $\mathbf{C}^*$-action of weight $-2m$, then the $(k-1)$-dimensional $\mathbf{C}^*$-module decomposes as a direct sum of 1-dimensional spaces with weights $-2(m-r)$ for $r = 1, \ldots, k-1$.*

Proof of 4.2.8: We shall give explicit Čech cohomology representatives for the weight vectors. We cover X with two charts $U_0 \times T$ and $U_\infty \times T$, where U_0, U_∞ are the standard affine neighbourhoods of 0 and ∞ in $\mathbf{P}^1$, and the transition map is given in the coordinates (z, t) on $U_0 \times T$ by

$$(z, t) \mapsto (z^{-1}, zt).$$

The bundle $\mathcal{O}_X(-k)$ is trivial over both $U_0 \times T$ and $U_\infty \times T$ with transition map given by

$$(z, t, \xi) \mapsto (z^{-1}, zt, z^k \xi).$$

On $U_0 \times T$ the $\mathbf{C}^*$-action is given by

$$\omega.(z, t) = (\omega^2 z, t),$$

and on $U_\infty \times T$ by

$$\omega.(\zeta, \tau) = (\omega^{-2}\zeta, \omega^2 \tau).$$

If we give $\mathcal{O}_X(-k)$ weight $-2m$, then the relation (4.1.4) shows that the monodromy of the action over T_∞ is given by $\mu = k - m$. Explicitly, the $\mathbf{C}^*$-action on $\mathcal{O}_X(-k)$ is given by

$$\begin{array}{rrcll} & \omega.(z,t,\xi) & = & (\omega^2 z, t, \omega^{-2m}\xi) & \text{over } U_0 \times T \\ \text{and} & \omega.(\zeta,\tau,\xi) & = & (\omega^{-2}\zeta, \omega^2\tau, \omega^{2(k-m)}\xi) & \text{over } U_\infty \times T \end{array}$$

Explicit representatives for the $(k-1)$-dimensional space $H^1(\mathcal{O}_X(-k))$ are given in $U_0 \times T$ coordinates on the intersection $(U_0 \times T) \cap (U_\infty \times T)$ by the functions $z^{-1}, \ldots, z^{-(k-1)}$. The action of $\mathbf{C}^*$ on a cocycle $f : (U_0 \setminus \{0\}) \times T \to \mathbf{C}$ in these coordinates is given by

$$(\omega.f)(z) = \omega^{-2m} f(\omega^{-2} z).$$

Hence, taking $f = z^{-r}$, we see that the subspace spanned by z^{-r} has weight $-2(m-r)$ for $r = 1, \ldots, k-1$. $\boxed{4.2.8}$

Corollary 4.2.9. *If $\mathcal{L}$ is a line bundle on X such that $\mathcal{L}^2 \cong \mathcal{O}_X(-k)$, where $0 > -k \geq -m$, and $\mathcal{L}$ has the $\mathbf{C}^*$-equivariant structure of weight $-m$, then given $\eta \in H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))$ representing the canonical class $1_Z \in \mathcal{O}_Z$, there is a unique projectively $\mathbf{C}^*$-invariant lift of η to $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$.*

Proof of 4.2.9: The local class η has weight 0 and lifts by Lemma 4.2.7. The space $H^1(\mathcal{O}_X(-k)$ has highest weight -2 by Lemma 4.2.8 and so no non-trivial elements of weight 0. Hence there is a unique projectively invariant lift. $\boxed{4.2.9}$

We can now prove:

Theorem 4.2.10. *Every monopole bundle representing a class in $\mathcal{M}(m, 1)$, with $m > 0$, is isomorphic to a $\mathbf{C}^*$-equivariant extension $\mathcal{E}$ of the form*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}\mathcal{J}_Z \to 0,$$

where $\mathcal{L}$ is a $\mathbf{C}^*$-equivariant bundle of weight $-m$ satisfying

(1) $-m < \deg \mathcal{L} < 0$, if $\mathcal{L}$ is generic;

(2) $-m/2 \leq \deg \mathcal{L} < 0$, if $\mathcal{L}$ is exceptional,

and $(Z, \mathcal{O}_Z)$ is the cluster supported at some point x_0 in T_0 whose ideal sheaf is generated in the local coordinates about x_0 defined above by $\{t, z^m\}$.

Conversely, given a $\mathbf{C}^*$-equivariant line bundle $\mathcal{L}$ satisfying (1) or (2) and a point $x_0 \in T_0$ with its associated cluster Z, there is a unique isomorphism class of stable $\mathbf{C}^*$-equivariant bundle $\mathcal{E}$ satisfying the above.

Proof of 4.2.10: It remains only to prove the last statement. Given the data Z and $\mathcal{L}$ as above, by Corollary 4.2.9 the canonical representative η in $H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L}))$ lifts uniquely to a projectively invariant element of $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Z, \mathcal{L})$, so defining a stable bundle $\mathcal{E}$ for which there are isomorphisms $\Phi_g : \mathcal{E} \to g^*\mathcal{E}$ for each $g \in \mathbf{C}^*$. We need to show that these isomorphisms fit together nicely to give an action of $\mathbf{C}^*$. Since $\mathcal{E}$ is stable, it is simple and the isomorphisms are uniquely defined up to scalars. However, we require the diagram

$$\begin{tikzcd}
0 \arrow[r] & g^*\mathcal{L} \arrow[r] & g^*\mathcal{E} \arrow[r] & (g^*\mathcal{L})^{-1}\mathcal{J}_Z \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[u, "\alpha_g"'] & \mathcal{E} \arrow[r] \arrow[u, "\Phi_g"'] & \mathcal{L}^{-1}\mathcal{J}_Z \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

to commute for each $g \in \mathbf{C}^*$; i.e., the isomorphism $\Phi_g : \mathcal{E} \to g^*\mathcal{E}$ must restrict to the given map $\alpha_g : \mathcal{L} \to g^*\mathcal{L}$. This fixes the choice of scalar uniquely. The uniqueness of the lifts Φ_g of the maps α_g and the relation (4.1.1) show that the Φ_g define an action of $\mathbf{C}^*$ on $\mathcal{E}$. Changing the choice of η by a non-zero scalar results in an isomorphic $\mathbf{C}^*$-bundle. $\boxed{4.2.10}$

Remark 4.2.11. Suppose $\mathcal{L}$ in the statement of Theorem 4.2.10 is exceptional. There are two possibilities for the type of $\mathcal{E}$ on restriction to a generic fibre of π, namely type (1) or (2). The first of these is ruled out by semi-continuity, since in this case $h^0(\pi^{-1}(x); \mathcal{L}\mathcal{E}) \geq 2$ for all $x \in \mathbf{P}^1$, but we know that $h^0(T_0, \mathcal{L}\mathcal{E}) = 1$. Hence

$$R^1\pi_*(\mathcal{L}^{-1}\mathcal{E}) \cong \mathcal{O}_{\mathbf{P}^1}(k) \oplus \mathcal{S},$$

where $\mathcal{S}$ is a skyscraper sheaf supported at $\infty \in \mathbf{P}^1$ and is non-zero if $\mathcal{E}|_{T_\infty}$ is of type (1). By Grothendieck-Riemann-Roch, we have $k + l(\mathcal{S}) = m$. An application of relative Serre duality then gives $\pi_*((\mathcal{L}^{-1}(-k))^{-1}\mathcal{E}) \cong \mathcal{O}_{\mathbf{P}^1}$ and so by the uniqueness of the maximal destabilising line bundle we have $\mathcal{L} \cong \mathcal{L}^{-1}(-k)$; i.e., $\mathcal{L}^2 \cong \mathcal{O}_X(-k)$. Hence, when $k = m$ we must have $E|_{T_\infty}$ of type (2). Otherwise the $\mathbf{C}^*$-equivariance forces $l(\mathcal{S}) \neq 0$ and the type of $\mathcal{E}$ jumps to type (1) on the fibre at ∞.

The monodromy of $\mathcal{E}$ around the path γ in T_∞ is given by an element $A \in \mathrm{SL}(2, \mathbf{C})$ with $\mathrm{Tr}(A) = \mu + \mu^{-1}$, where μ, μ^{-1} are the corresponding monodromies of the maximal destabilising line bundles of $\mathcal{E}$. Given $\mathrm{Tr}(A)$, the unordered pair $\{\mu, \mu^{-1}\}$ of eigenvalues of A is uniquely determined and consequently $\mathrm{Tr}(A)$ uniquely determines the pair $\{\lambda^{-m/2}\mu, \lambda^{-m/2}\mu^{-1}\}$ of maximal destabilising line bundles in $\mathbf{C}^* \cong \mathrm{Pic}(X)$. From Theorem 4.2.10 we see the pair $\{\lambda^{-m/2}\mu, \lambda^{-m/2}\mu^{-1}\}$ determines uniquely a point in the quotient $\mathbf{A}/\sim$ where $\mathbf{A} \subset \mathrm{Pic}(X) \cong \mathbf{C}^*$ is the subset

$$\mathbf{A} = \{\mathcal{L} \in \mathrm{Pic}(X) : \frac{-(m+1)}{2} < \deg \mathcal{L} < 0\} = \{\mu \in \mathbf{C}^* : \lambda^{-(m+1)/2} < |\mu| < 1\}.$$

and $\sim$ is the equivalence relation generated by $\nu \sim \lambda^{-m}\nu$. Let $\mathbf{A}_m \subset \mathrm{Pic}(X)$ denote the annulus

$$\mathbf{A}_m = \{\mathcal{L} \in \mathrm{Pic}(X) : -m < \deg \mathcal{L} < 0\} = \{\mu \in \mathbf{C}^* : \lambda^{-m} < |\mu| < 1\}$$

There is a $\mathbf{Z}_2$-action on $\mathbf{A}_m$ generated by $\mu \mapsto \mu^{-1}\lambda^{-m}$. The inclusion $\mathbf{A} \to \mathbf{A}_m$ induces an isomorphism

$$(\mathbf{A}/\sim) \cong \mathbf{A}_m/\mathbf{Z}_2.$$

As in the proof of (3.7.5), the latter space is isomorphic to $\mathbf{P}^1 \setminus I$, where I is a copy of the closed interval $[0,1]$, and this in turn is diffeomorphic to D^2. Hence, taking the trace of the monodromy of $\mathcal{E} \in \mathcal{M}(m,1)$ around γ gives a holomorphic map

$$\mathrm{Tr} : \mathcal{M}(m,1) \to \mathbf{A}_m/\mathbf{Z}_2.$$

Restriction of $\mathcal{E}$ to the fibre T_0 gives a holomorphic map

$$r : \mathcal{M}(m,1) \to \mathrm{Pic}^1(T_0) \cong T_0.$$

A point of $\mathbf{A}_m$ uniquely determines maximal destabilising line bundles $\mathcal{L}_1, \mathcal{L}_2$ satisfying the conditions of Theorem 4.2.10. A choice of $\mathcal{L} \in \mathrm{Pic}^1(T_0)$ then determines the corresponding clusters Z_1, Z_2 uniquely and hence by Theorem 4.2.10 a unique isomorphism class of $\mathbf{C}^*$-equivariant $\mathrm{SL}(2,\mathbf{C})$ bundle $\mathcal{E}$. Hence we have a bijective holomorphic map

$$\mathrm{Tr} \times r : \mathcal{M}(m,1) \to (\mathbf{A}/\sim) \times \mathrm{Pic}^1(T_0)$$

leading to:

Theorem 4.2.12. *The moduli space $\mathcal{M}(m,1)$ is diffeomorphic to $D^2 \times T_0$.*

Remark 4.2.13. This result was also obtained by Braam and Hurtubise by their patching arguments [16, Theorem 4.5.1].

Chapter 5

Higher strata

In this chapter we study bundles in the higher strata $\mathcal{M}_k^r$ where $1 \leq r \leq k$. These are the bundles $\mathcal{E}$ the graph part of whose divisor is that of a non-constant rational map $\alpha : \mathbf{P}^1 \rightarrow \mathbf{P}^1$. In particular this means that there can be no line bundle on X admitting a non-trivial map to $\mathcal{E}$, so $\mathcal{E}$ is automatically stable. Another consequence is that the bundles cannot be constructed by the method of Serre.

For $\mathrm{SL}(2, \mathbf{C})$ bundles over a regular algebraic elliptic surface Friedman [24] developed a construction first used by Brosius for ruled surfaces [17], which with few modifications can also be used for our Hopf surface. The main problem is that the Picard group of the Hopf surface X is not "big enough" to supply us with enough line bundles to use the Serre construction. However, the graphs of any $\mathcal{E} \in \mathcal{M}_k^r$ for $1 \leq r \leq k$ define naturally a double cover $\bar{\nu} : \bar{X} \longrightarrow X$ on which the pull-back $\bar{\mathcal{E}} = \bar{\nu}^* \mathcal{E}$ may be constructed by the method of Serre; i.e., there is a line bundle $\mathcal{L}$ and a cluster $(Z, \mathcal{O}_Z)$ on $\bar{X}$ such that $\bar{\mathcal{E}}$ fits into an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{L} \longrightarrow \bar{\mathcal{E}} \longrightarrow \mathcal{L}^{-1} \mathcal{J}_Z \longrightarrow 0.$$

In the generic case, pushing down $\mathcal{L}$ by $\bar{\nu}$ results in a rank 2 bundle $\mathcal{F}$ on X, which when twisted by a line bundle gives $\mathcal{E}$. Friedman uses algebraic geometry to deduce the existence of $\mathcal{L}$ and its subsequent properties. Since X is non-algebraic we must use other methods.

5.1. Graphs and double covers

Let $\mathcal{E}$ be a bundle on X defining a class in $\mathcal{M}_k^r$ and whose divisor $D(\mathcal{E})$ is given by

5.1.1. $$D(\mathcal{E}) = G_\alpha + \sum_{i=1}^{k-r} F_i,$$

where $\alpha : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is a degree r rational map and $\{F_i : i = 1, \ldots, k-r\}$ is a collection of reduced fibres of $\pi : X \rightarrow \mathbf{P}^1$ (not all necessarily distinct). Let

$$\bar{q} : \mathbf{P}^1 \times T \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

denote the double cover induced by the hyperelliptic quotient $q : T \to \mathbf{P}^1$. Projection onto the first factor $\pi_1 : \mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ defines an isomorphism of the curve $G_\alpha \subset \mathbf{P}^1 \times \mathbf{P}^1$ with $\mathbf{P}^1$. The pre-image of G_α under $\bar{q}$ is a divisor C in $\mathbf{P}^1 \times T$. By construction, a point $(x, \mu) \in \mathbf{P}^1 \times T$ lies on C if and only if $\mathcal{E}|_{\pi^{-1}(x)}$ admits $\mathcal{V}_\mu|_{\pi^{-1}(x)}$ as a sub-bundle. If $B' = \pi_{1*}(G_\alpha . H)$, where H is the horizontal $(4, 0)$ divisor on $\mathbf{P}^1 \times \mathbf{P}^1$ given by

$$H = \cup_{l = q(\text{half-period})} \mathbf{P}^1 \times \{l\},$$

then $\bar{q}$ restricted to C defines a double cover

$$\pi_1 \circ \bar{q} : C \longrightarrow \mathbf{P}^1,$$

with branch locus given by the degree 4r divisor B'. The curve C may, however, have singularities which, since G_α is smooth, can only occur at ramification points; i.e., points $(x, \mu) \in C \cap H$. More precisely we have:

Lemma 5.1.2. *A point $(x, \mu) \in C$ is singular if and only if $(x, \mu) \in C \cap H$ and the line $\mathbf{P}^1 \times q(\mu) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is tangent to G_α at $(x, \alpha(x)) = (x, q(\mu))$.*

Proof of 5.1.2: Taking local coordinates x and t on $\mathbf{P}^1$ and T respectively, we may regard α and $\bar{q}$ locally as functions. The divisor C is then the curve cut out locally by the equation

5.1.3. $$\phi(x, t) \stackrel{\text{def}}{=} \alpha(x) - \bar{q}(t) = 0.$$

The derivative

$$D\phi_{(x,\mu)}(\xi, \tau) = D\alpha_x(\xi) - Dq_\mu(\tau)$$

vanishes at $(x, \mu) \in C$ if and only if both Dq_μ and $D\alpha_x$ vanish. The first condition is precisely that μ be a point of ramification of q, and the second shows that the line $\mathbf{P}^1 \times \{\mu\}$ is tangent to G_α at $(x, \alpha(x))$. $\boxed{5.1.2}$

Lemma (5.1.2) shows that we expect C generically to be smooth. If not, we may blow up $\mathbf{P}^1 \times T$ succesively at the singular points of C until the proper transform Σ of C is a smooth curve. The natural map $\Sigma \to C$ is then a resolution of singularities. In either case, we obtain a double cover

$$\nu : \Sigma \longrightarrow \mathbf{P}^1$$

with branch locus a subset of the set $B' \subset \mathbf{P}^1$. The genus of Σ depends on the local intersection numbers of the graph of the map $\alpha : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ and the divisor H at points of intersection.

Let $(x, \alpha(x))$ be a point of intersection of G_α and H. We may choose local coordinates z on $\mathbf{P}^1$ and t on T centred on x and $\alpha(x)$ respectively such that locally C is given by the set of $(z, t) \in \mathbf{P}^1 \times T$ satisfying the equation

5.1.4. $$t^2 - z^m = 0,$$

where m is the local intersection number of $\mathbf{P}^1 \times \{\alpha(x)\}$ and G_α at $(x, \alpha(x))$. The singularity is therefore a double point and there are two distinct cases:

(a) $m = 2n$ is even;

(b) $m = 2n + 1$ is odd.

In each case the singularity may be resolved by blowing up n times at the point $(x, \alpha(x))$ [32, p.393]. The curve Σ is then given in local coordinates (z, u) on the blown-up surface in each case by (a) $u^2 = 1$, and (b) $u^2 = z$. In (a) the curve C has two branches locally, given by $z^n - t = 0$ and $z^n + t = 0$. These are "pulled apart" under the resolution. A local coordinate for each branch is given by z. In case (b) the curve is locally irreducible and has a cusp singularity. A local coordinate on Σ is given by u. The resolution $\Sigma \to C$ is given locally in each case by (a) $z \mapsto (z, z^n)$, and (b) $u \mapsto (u^2, u^m)$. The pull-back in $\Sigma \times T$ of the divisor $C \subset \mathbf{P}^1 \times T$ via the double cover $\nu : \Sigma \to \mathbf{P}^1$ then reduces locally with two branches given by the equations: (a) $t \pm z^n = 0$; (b) $t \pm u^m = 0$.

Let N_0, N_1 denote the number of points $(z_i, t_i) \in C \cap \bar{q}^{-1}(H)$ near which C is given locally by (5.1.4) with m odd or even respectively. Suppose further that for $i = 1, \ldots, N_0$, (z_i, t_i) is given by (5.1.4) with $m = m_i = 2n_i + 1$, and for $i = N_0 + 1, \ldots, N_0 + N_1$, (z_i, t_i) is given by $m = m_i = 2n_i$. Setting $N = N_0 + N_1$, we have:

Lemma 5.1.5. *The following equalities hold:*

(a) $N_0 = 4r - 2\sum_{i=1}^N n_i$;

(b) The genus $g(\Sigma)$ of Σ is given by:

$$g(\Sigma) = \frac{1}{2}(N_0 - 2) = 2r - \sum_{i=1}^N n_i - 1.$$

Proof of 5.1.5: We have $\sum_{i=1}^N m_i = G_\alpha.H = 4r$. This gives $2\sum_{i=1}^N n_i + N_0 = 4r$. The curve Σ is a 2:1 cover of $\mathbf{P}^1$ branched over the points $z_i, i = 1, \ldots, N_0$. Hence, using the Riemann-Hurwitz formula [30, p.219]

$$2 - 2g(\Sigma) = \chi(\Sigma) = 2(\chi(\mathbf{P}^1) - N_0) + N_0,$$

we obtain the result. $\boxed{5.1.5}$

Thus to any $\mathrm{SL}(2, \mathbf{C})$ bundle $\mathcal{E} \in \mathcal{M}_k^r$ with $r > 0$ we can associate in a canonical way a smooth curve Σ and a double cover $\nu : \Sigma \to \mathbf{P}^1$.

Remark 5.1.6. If $g(\Sigma) \geq 2$ then Σ is hyperelliptic.

Let $\tilde{X} = \nu^* X$ be the principal elliptic surface obtained by pulling back X to Σ. We have a commutative diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\nu}"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\nu"] & \mathbf{P}^1 \end{tikzcd}$$

where the map $\tilde{\nu}$ is both a double cover and a morphism of principal elliptic bundles. Let B denote the branch locus of ν, so $B = \{x_1, \ldots, x_{N_0}\}$. If $y_i = \nu^{-1}(x_i)$, then the ramification divisor R is given by $R = \{y_1, \ldots, y_{N_0}\}$. The branch locus of $\tilde{\nu}$ is then given by

$$\overline{B} = \sum_{i=1}^{N_0} \pi^{-1}(x_i),$$

and the ramification divisor by

$$\overline{R} = \sum_{i=1}^{N_0} \tilde{\pi}^{-1}(y_i).$$

On Σ there is a holomorphic involution interchanging the two sheets, which lifts naturally to $\tilde{X}$. We shall denote both involutions by ι.

Next we recall some facts about the geometry of double covers, referring to [12] and [32, p.306] for further details. Since $\mathrm{Pic}(\mathbf{P}^1)$ is torsion free, the double cover Σ is unique up to isomorphism. The push-down $\nu_*\mathcal{O}_\Sigma$ is a locally free rank 2 sheaf on $\mathbf{P}^1$ and decomposes:

$$\nu_*\mathcal{O}_\Sigma \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-nP),$$

where P is the divisor class corresponding to a point on $\mathbf{P}^1$ and the integer n satisfies $2n = \deg(B)$. The two summands $\mathcal{O}_{\mathbf{P}^1}$ and $\mathcal{O}_{\mathbf{P}^1}(-nP)$ correspond to the $+1$ and -1 eigenspaces of the natural action of the involution ι on $\nu_*\mathcal{O}_\Sigma$ respectively. Moreover, if Q is a divisor on Σ, the sheaf $\nu_*\mathcal{O}_\Sigma(Q)$ is locally free and we have

5.1.7. $$\det \nu_*\mathcal{O}_\Sigma(Q) \cong \mathcal{O}_{\mathbf{P}^1}(\nu_*Q - nP).$$

The ramification divisor R of the double cover satisfies

$$\overline{R} \sim n(\nu^*P).$$

Denoting the divisor $\frac{1}{2}\deg \overline{B}(\pi^*P)$ on X by D and using the commutative diagram of double covers we therefore obtain

5.1.8. $$2\tilde{\nu}^*D \sim R.$$

Finally, since $\tilde{X}$ is non-algebraic, a generic line bundle $\mathcal{L}$ on $\tilde{X}$ does not correspond to a divisor. (In fact all effective divisors on $\tilde{X}$ are sums of fibres of $\tilde{\pi}$). There is therefore no exact analogue of (5.1.7) for the rank 2 bundle $\tilde{\nu}_*\mathcal{L}$ on X. However, we do have the following, the proof of which is a tedious check of transition functions.

Lemma 5.1.9. *If $\mathcal{L}$ is a line bundle on $\tilde{X}$ then the line bundle* $\det \tilde{\nu}^*\tilde{\nu}_*\mathcal{L}$ *is isomorphic to $\mathcal{L} \otimes \iota^*\mathcal{L} \otimes \mathcal{O}_{\tilde{X}}(-R)$.*

5.2. The Serre construction on the double cover

Let $\mathcal{E}$ be a fixed SL(2, $\mathbf{C}$) bundle on X with a divisor of the form (5.1.1) and let $\tilde{\nu} : \tilde{X} \to X$ be the double cover constructed in (5.1). Let $\tilde{\mathcal{E}} = \tilde{\nu}^*\mathcal{E}$ so $\tilde{\mathcal{E}} \in \mathrm{SL}_0(\tilde{X})$ and has a divisor $\tilde{D} = D(\tilde{\mathcal{E}})$, given by the pull-back of $D(\mathcal{E})$:

$$\tilde{D} = G_{\alpha\circ\nu} + \sum_{i=1}^{k-r} \tilde{\nu}^* F_i,$$

where $G_{\alpha\circ\nu}$ denotes the graph of the holomorphic map

$$\alpha \circ \nu : \Sigma \longrightarrow \mathbf{P}^1.$$

Under the quotient map $\Sigma \times T \to \Sigma \times \mathbf{P}^1$, $G_{\alpha\circ\nu}$ is double-covered by $\tilde{G}$ where

$$\begin{aligned}\tilde{G} &= \{(x,t) \in \Sigma \times T : \alpha(\nu(x)) = q(t)\}\\ &= \{(x,t) \in \Sigma \times T : (\nu(x), t) \in C\}.\end{aligned}$$

We now come to the motivation for the introduction of the double cover.

Lemma 5.2.1. *The divisor $\tilde{G}$ in $\Sigma \times T$ is reducible and decomposes:*

$$\tilde{G} = G_\gamma + \iota^* G_\gamma,$$

where $\iota : \Sigma \times T \longrightarrow \Sigma \times T$ is the holomorphic involution generating the $\mathbf{Z}_2$-action and G_γ is the graph of the tautological map γ defined by the composition

$$\Sigma \xrightarrow{\nu} C \hookrightarrow \mathbf{P}^1 \times T \xrightarrow{\pi_2} T.$$

By lemma (2.2.8) there are line bundles $\mathcal{L}_0, \mathcal{L}_0'$ in $\mathrm{Pic}^0(\tilde{X}; \Sigma)$ with graphs $\gamma, \iota^*\gamma$ respectively. We then obtain the following:

Proposition 5.2.2. *The bundle $\tilde{\mathcal{E}}$ fits into two distinct exact sequences of sheaves on $\tilde{X}$,*

5.2.3.
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{L}_1 & \to & \tilde{\mathcal{E}} & \to & \mathcal{L}_1^{-1}\mathcal{J}_{Z_1} & \to & 0; \\ 0 & \to & \mathcal{L}_2 & \to & \tilde{\mathcal{E}} & \to & \mathcal{L}_2^{-1}\mathcal{J}_{Z_2} & \to & 0, \end{array}$$

where $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}^0(\tilde{X}, \Sigma)$ have graphs $\gamma, \iota^*\gamma$ respectively, and Z_1, Z_2 are clusters on $\tilde{X}$. They satisfy $\iota^*\mathcal{L}_1 \cong \mathcal{L}_2$ and $\iota^* Z_1 = Z_2$. We also have

$$l(Z_i) - c_1(\mathcal{L}_i)^2 = 2k, \ \textit{for } i = 1, 2.$$

Proof of 5.2.2: The existence of both sequences follows by (2.4.12). Taking $\mathrm{Ext}^*(\mathcal{L}_1, \cdot)$ of the first sequence gives an exact sequence

$$0 \to h^0(\mathcal{O}_{\tilde{X}}) \to H^0(\mathcal{L}_1^{-1}\tilde{\mathcal{E}}) \to H^0(\mathcal{L}_1^{-2}\mathcal{J}_{Z_1}).$$

Since $\mathcal{L}_1^2$ is non-trivial on restriction to a generic fibre of $\tilde{\pi}$ the last term vanishes, so the map $\mathcal{L}_1 \to \tilde{\mathcal{E}}$ is determined uniquely up to scalar multiplication. Consequently, the cluster Z_1 is uniquely determined. Taking ι^* of the first sequence gives the second sequence, by uniqueness. The formula relating degrees follows by applying the Chern character formula to each exact sequence. $\boxed{5.2.2}$

Thus although the bundle $\mathcal{E}$ may not be constructed by the method of Serre, its pull-back $\tilde{\mathcal{E}}$ to $\tilde{X}$ can.

5.3. The Friedman construction

For the moment denote $\mathcal{L}_1$ and Z_1 in the exact sequences of Proposition 5.2.2 simply by $\mathcal{L}$ and Z. The rank 2 sheaf $\tilde{\nu}_*\mathcal{L}$ is locally free and defines a rank 2 bundle on X, which by construction is isomorphic to $\mathcal{E}$ on generic fibres of π. Pushing down the exact sequence

5.3.1. $$0 \to \mathcal{L} \to \bar{\mathcal{E}} \to \mathcal{L}^{-1}\mathcal{J}_Z \to 0,$$

results in an inclusion

$$0 \to \tilde{\nu}_*\mathcal{L} \to \tilde{\nu}_*\bar{\mathcal{E}}.$$

Applying the projection formula $\tilde{\nu}_*\tilde{\nu}^*\mathcal{E} \cong \mathcal{E} \otimes \tilde{\nu}_*\mathcal{O}_{\tilde{X}}$ [32, p.124] to the final term, we have an injective map of *sheaves*:

5.3.2. $$0 \to \nu_*\mathcal{L} \to \mathcal{E} \oplus \mathcal{E}(-D).$$

The following lemma is due to Friedman.

Lemma 5.3.3 (Friedman, 5.4). *The natural maps*

$$\tilde{\nu}_*\mathcal{L} \to \mathcal{E};$$
$$\tilde{\nu}_*\mathcal{L} \to \mathcal{E}(-D),$$

obtained by projection onto each factor in (5.3.2) are both non-zero.

Proof of 5.3.3: The decomposition (5.3.2) is obtained by taking the ± 1 eigenspaces for the action of ι on $\tilde{\nu}_*\bar{\mathcal{E}}$. The sheaf $\tilde{\nu}_*\mathcal{L}$ is a $\tilde{\nu}_*\mathcal{O}_{\tilde{X}}$-module and contains -1 eigenvectors for the ι-action. If the image of $\tilde{\nu}_*\mathcal{L}$ were contained entirely within $\mathcal{E}$, then choosing any non-zero element of $(\tilde{\nu}_*\mathcal{L})_x$ and multiplying it by a -1 eigenvector in $(\tilde{\nu}_*\mathcal{O}_{\tilde{X}})_x$, for any $x \in X$, gives a non-zero element of $(\tilde{\nu}_*\mathcal{L})_x$ which maps to $\mathcal{E}(-D)$ under the map (5.3.2). Similarly, the component of the map in the other factor cannot vanish. $\boxed{5.3.3}$

Proposition 5.3.4. *The map $\phi : \tilde{\nu}_*\mathcal{L} \to \mathcal{E}(-D)$ is an injection of sheaves.*

Proof of 5.3.4: The map is non-zero by (5.3.3) and so has rank 1 or 2 generically. Let $\mathcal{R}$ be the kernel, so we have an exact sequence of sheaves

$$0 \to \mathcal{R} \to \tilde{\nu}_*\mathcal{L} \xrightarrow{\phi} \mathcal{E}(-D).$$

The last two sheaves are locally free so $\mathcal{R}$ is locally a 2nd syzygy sheaf and hence locally free, with rank either 0 or 1, outside a set of codimension 3; i.e., on all of X.

If $\mathrm{rank}(\mathcal{R}) = 1$, then $\mathcal{R}$ is a line bundle on X and its graph is that of a constant map $\mathbf{P}^1 \to \mathbf{P}^1$. However, the divisor of $\tilde{\nu}_*\mathcal{L}$ is that of a non-constant rational map $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$, giving a contradiction. Hence, $\mathrm{rank}(\mathcal{R}) = 0$ and the map ϕ is an inclusion of sheaves.

Hence we have an inclusion

$$(\tilde{\nu}_*\mathcal{L}) \otimes \mathcal{O}_X(D) \to \mathcal{E},$$

i.e.,

$$\tilde{\nu}_*(\mathcal{L} \otimes \tilde{\nu}^*\mathcal{O}_X(D)) = \tilde{\nu}_*(\mathcal{L}(R)) \to \mathcal{E}.$$

Writing $\mathcal{F} = \tilde{\nu}_*(\mathcal{L}(R))$ and using $R \sim \tilde{\nu}^*D$, we have a short exact sequence of sheaves

5.3.5. $$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where the quotient sheaf $\mathcal{Q}$ is a torsion sheaf supported on the divisor on X where the map of *bundles* $\mathcal{F} \to \mathcal{E}$ fails to have full rank; i.e., on the zero-divisor F of the induced map of determinant line bundles $\det \mathcal{F} \to \det \mathcal{E}$.

The bundle $\mathcal{L} \otimes (\iota^*\mathcal{L})$ is ι-invariant and hence is the pull-back of some line bundle on X. Thus,

5.3.6. $$\mathcal{L} \otimes (\iota^*\mathcal{L}) \cong \mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{O}_{\tilde{X}}(-M),$$

for some divisor M supported in the fibres of $\tilde{\pi}$. Pulling back the exact sequence (5.3.5) to $\tilde{X}$ gives:

5.3.7. $$0 \to \tilde{\mathcal{F}} \to \tilde{\mathcal{E}} \to \tilde{\mathcal{Q}} \to 0$$

where $\tilde{\mathcal{F}} = \tilde{\nu}^*\mathcal{F}$ and $\tilde{\mathcal{Q}} = \tilde{\nu}^*\mathcal{Q}$. An application of Lemma 5.1.9 to $\mathcal{L}(R)$ using (5.3.6) shows the torsion sheaf $\tilde{\mathcal{Q}}$ is supported on the divisor

5.3.8. $$\tilde{F} = \tilde{\nu}^*F \sim M - R.$$

Since the divisor $\tilde{F}$ is effective, it must consist of a sum of fibres of $\tilde{\pi}$.

The natural surjection

$$\tilde{\nu}^*\tilde{\nu}_*(\mathcal{L}(R)) \to \mathcal{L}(R) \to 0$$

gives rise to an exact sequence

$$0 \to \mathcal{L}^{-1}(-M) \to \tilde{\mathcal{F}} \to \mathcal{L}(R) \to 0.$$

Applying ι^* to this sequence and using (5.3.6) and (5.3.8) gives

5.3.9. $$0 \to \mathcal{L} \to \tilde{\mathcal{F}} \to \mathcal{L}^{-1}(-\tilde{F}) \to 0.$$

The exact sequences (5.3.1), (5.3.7) and (5.3.9) fit together as described in the following lemma of Friedman:

Lemma 5.3.10 (Friedman, 5.6). *For $i = 1, 2$ there is a commutative diagram of exact sequences of sheaves:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{L}_i \arrow[r, equal] \arrow[d] & \mathcal{L}_i \arrow[d] & & \\
0 \arrow[r] & \bar{\mathcal{F}} \arrow[r] \arrow[d] & \bar{\mathcal{E}} \arrow[r] \arrow[d] & \bar{Q} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}_i^{-1}(-\bar{F}) \arrow[r] \arrow[d] & \mathcal{L}_i^{-1}\mathcal{J}_{Z_i} \arrow[r] \arrow[d] & \bar{Q} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

The two diagrams are interchanged by an application of ι^*.

Corollary 5.3.11. *For $i = 1, 2$ there is an inclusion $\mathcal{O}_{\tilde{X}}(-\bar{F}) \hookrightarrow \mathcal{J}_{Z_i}$ and an isomorphism of sheaves:*

$$\bar{Q} \cong \{\mathcal{J}_{Z_i}/\mathcal{O}_{\tilde{X}}(-\bar{F})\} \otimes \mathcal{L}_i^{-1}.$$

From the second column of the diagram it is clear that on each fibre f of $\tilde{\pi}$ where the Z_i are supported, $\bar{\mathcal{E}}|_f$ is of type (3). By construction, the divisor and graph of $\mathcal{F}$ are equal and identical to the graph of $\mathcal{E}$. Hence $\mathcal{F}$ restricts to a bundle of type (1) or (2) on the fibres of $\tilde{\pi}$. Consequently, the support of $\bar{Q}$ must contain those fibres f which contain a point of the Z_i and those fibres of R where $\bar{\mathcal{F}}$ and $\bar{\mathcal{E}}$ are extensions of half-periods of different types.

The 'size' of the divisor $\bar{F}$ is estimated as follows. Taking $\mathrm{Ext}^*(\mathcal{L}_2, \cdot)$ of the first and second columns of the commutative diagram in 5.3.10 with $i = 1$ and using the relation

$$(\mathcal{L}_1\mathcal{L}_2)^{-1} \cong \mathcal{O}_{\tilde{X}}(M) \cong \mathcal{O}_{\tilde{X}}(R + \bar{F}),$$

gives a commutative diagram of long exact sequences:

5.3.12.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{L}_2^{-1}\bar{\mathcal{E}}) \arrow[r] & H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F})\mathcal{J}_{Z_1}) \arrow[r, "\delta"] & H^1(\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R+\bar{F})) \arrow[r] & {} \\
0 \arrow[r] & H^0(\mathcal{L}_2^{-1}\bar{\mathcal{F}}) \arrow[r] \arrow[u] & H^0(\mathcal{O}_{\tilde{X}}(R)) \arrow[r] \arrow[u] & H^1(\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R)) \arrow[r] \arrow[u] & {} \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\mathcal{L}_2$ is a maximal destabilising line bundle for $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$,

$$\dim H^0(\mathcal{L}_2^{-1}\bar{\mathcal{E}}) = \dim H^0(\mathcal{L}_2^{-1}\bar{\mathcal{F}}) = 1.$$

Hence $H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F})\mathcal{J}_{Z_1}) \neq 0$ and a generator of $H^0(\mathcal{L}_2^{-1}\mathcal{F})$ defines via the diagram (5.3.12) an inclusion

5.3.13. $$\mathcal{O}_{\tilde{X}}(-R-\bar{F}) \hookrightarrow \mathcal{J}_{Z_1},$$

which is the image in $H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F})\mathcal{J}_{Z_1})$ of an element of $H^0(\mathcal{O}_{\tilde{X}}(R))$. If f is a fibre of $\tilde{\pi}$ contained in the support of $\bar{F}$ and such that the map 5.3.13 factors

5.3.14. $$\mathcal{O}_{\tilde{X}}(-R-\bar{F}) \hookrightarrow \mathcal{O}_{\tilde{X}}(-R-\bar{F}+rf) \hookrightarrow \mathcal{J}_{Z_1},$$

for some $r > 0$, then we obtain a non-zero section σ in $H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F}-rf)\mathcal{J}_{Z_1})$. We may think of this space as the global sections of $\mathcal{O}_{\tilde{X}}(R+F-rf)$ which vanish at the points in the support of Z_1 in the manner prescribed by the sheaf $\mathcal{J}_{Z_1}$. Clearly, for r large enough the space of such sections is trivial. Under the composition of the natural inclusion

$$H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F}-rf)\mathcal{J}_{Z_1}) \to H^0(\mathcal{O}_{\tilde{X}}(R+\bar{F})\mathcal{J}_{Z_1}),$$

and the coboundary map δ of 5.3.12, the map σ defines an element of $H^1(\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R+\bar{F}))$. The last space is isomorphic via the Leray spectral sequence to $H^0(R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R+\bar{F}))$. As we increase r and therefore "decrease the size" of the divisor $R+\bar{F}-rf$, we obtain a family of global sections of $R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R+\bar{F})$. Since $H^0(R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(R+\bar{F}))$ is finite dimensional, we therefore obtain a bound on the size of $\bar{F}$. This bound was calculated by Friedman and the following lemma, whose proof consists of a local analysis, comes directly from his paper.

Lemma 5.3.15 (Friedman, Lemma 5.8). *Suppose $f = \tilde{\pi}^{-1}(x)$ is a fibre of $\tilde{\pi}$ above $x \in \Sigma$ for which we have a factorisation*

$$\mathcal{O}_{\tilde{X}}(-\bar{F}) \hookrightarrow \mathcal{O}_{\tilde{X}}(-\bar{F}+rf) \longrightarrow \mathcal{J}_{Z_1},$$

where the first map is the natural inclusion, then

$$r \leq \begin{cases} l(R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(M)_x), & \text{if } f \not\subset R; \\ l(R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(M)_x) - 1, & \text{if } f \subset R. \end{cases}$$

Note that if $x \notin \{y_i : i = 1, \ldots, N\}$ then $l((R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(M))_x) = 0$ and as a result of 5.3.15 (using 3.4.3) we obtain:

Corollary 5.3.16. *The support of $\bar{F}$ does not contain any fibres of $\tilde{\pi}$ over $\Sigma \setminus \{y_i : i = 1, \ldots, N\}$ on which $\mathcal{E}$ is of type (1). Furthermore, if $x \neq y_i$ for $i = 1, \ldots, N$ and on $\tilde{\pi}^{-1}(x)$ the bundle $\mathcal{E}$ is of type (3), then the fibre $f = \tilde{\pi}^{-1}(x)$ is a component of $\bar{F}$ of multiplicity precisely $w_f(Z_1)$.*

The lemma furthermore gives bounds for the size of $\bar{F}$ at other fibres. The precise value for these bounds is given by the next lemma.

Lemma 5.3.17. *For $i = 1, \ldots, N$ we have:*

$$l(R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(M)_{w_i}) = \begin{cases} m_i, & 1 \leq i \leq N_0; \\ n_i, & N_0 + 1 \leq i \leq N. \end{cases}$$

Proof of 5.3.17: Recall from (2.2.8) that $\mathcal{L}_1 \cong \mathcal{L} \otimes \tilde{\pi}^*\mathcal{V}$ for some line bundle $\mathcal{V}$ on Σ, where $\mathcal{L}$ was constructed by patching together bundles $\mathcal{L}_\alpha$ defined over local trivialisations $U_\alpha \times T$ for $\tilde{X}$. By the projection formula, the skyscraper sheaves $R^1\tilde{\pi}_*\mathcal{L}_1^2\mathcal{O}_{\tilde{X}}(M)$ and $R^1\tilde{\pi}_*\mathcal{L}^2$ are isomorphic. Since the problem is local, we may restrict to a trivialisation $U_\alpha \times T$ where $y_i \in U_\alpha$ and $\mathcal{L} \cong \mathcal{L}_\alpha$. By definition, the line bundle $\mathcal{L}_\alpha$ is the pull-back of a fixed Poincaré bundle $\mathcal{P}$ on $\mathrm{Pic}^0(T) \times T$ by the map:

$$\begin{aligned} \gamma_\alpha : U_\alpha \times T &\longrightarrow \mathrm{Pic}^0(T) \times T \\ (z, t) &\longrightarrow (\gamma(z), t), \end{aligned}$$

where γ is the graph of $\mathcal{L}_1$. A local resolution

$$0 \longrightarrow \mathcal{O}_{U_\alpha}^m \xrightarrow{A} \mathcal{O}_{U_\alpha}^m \longrightarrow R^1\pi_{1*}\mathcal{P}^2 \longrightarrow 0,$$

(where A is a matrix of functions holomorphic on U_α) for $R^1\pi_{1*}\mathcal{P}^2$ on U_α then pulls-back via the map γ to a local resolution for $R^1\tilde{\pi}_*\mathcal{L}_\alpha^2$. By the definition of the first Chern class for coherent sheaves [34, Chapter 5], $c_1(R^1\pi_{1*}\mathcal{P}^2)$ is given locally on U_α by the zero-divisor of $\det(A \circ \gamma)$. There is a local coordinate z on U_α for which y_i is given by $z = 0$ and the map γ is given locally by

$$z \mapsto \begin{cases} z^{m_i}, & i = 1, \ldots, N_0; \\ z^{n_i}, & i = N_0 + 1, \ldots, N. \end{cases}$$

It then follows that

$$l((R^1\tilde{\pi}_*\mathcal{L}_\alpha^2)_{w_i}) = \begin{cases} m_i . l((R^1\pi_{1*}\mathcal{P}^2)_{\gamma(w_i)}), & i = 1, \ldots, N_0; \\ n_i . l((R^1\pi_{1*}\mathcal{P}^2)_{\gamma(w_i)}), & i = N_0 + 1, \ldots, N. \end{cases}$$

Claim (to be proved in (5.3.18)): If $\mu \in \mathrm{Pic}^0(T)$ is a half-period, then $l((R^1\pi_{1*}\mathcal{P}^2)_\mu) = 1$.

The result then follows from this claim. $\boxed{5.3.17}$

Lemma 5.3.18. *Let $\mathcal{P} \to \mathrm{Pic}^0(T) \times T$ be a Poincaré bundle and let*

$$\pi_1 : \mathrm{Pic}^0(T) \times T \to \mathrm{Pic}^0(T)$$

be projection, then $R^1\pi_{1*}\mathcal{P}^2$ is a skyscraper sheaf on $\mathrm{Pic}^0(T)$ supported at the four half-periods, and for each half-period μ we have $l((R^1\pi_{1*}\mathcal{P}^2)_\mu) = 1$.

Proof of 5.3.18: Without loss of generality we may take $\mathcal{P}$ to be the Poincaré line bundle given by the quotient of $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$ by the $\mathbf{Z} \times \mathbf{Z}$ action generated by:

$$\begin{aligned} (1, 0).(\mu, z, \xi) &= (\mu, \lambda z, \mu\xi); \\ (0, 1).(\mu, z, \xi) &= (\lambda\mu, z, z\xi). \end{aligned}$$

The metric $\|\cdot\|$ on the trivial line bundle $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$ defined on sections s by

$$\|s(\mu, z)\| = |s(\mu, z)|.|z|^{\log|\mu|/\log\lambda}.|\mu|^{\log|z|/\log\lambda},$$

where $|\cdot|$ is the standard metric on $\mathbf{C}$, is $\mathbf{Z} \times \mathbf{Z}$ equivariant and so descends to $\mathcal{P}$. The curvature form of the associated Hermitian connection A is then given by

$$F_A = 2\bar{\partial}\partial\{\frac{\log|\mu|\log|z|}{\log\lambda}\},$$

giving

$$F_A \wedge F_A = -\frac{d\mu \wedge d\bar{\mu} \wedge dz \wedge d\bar{z}}{2(\log\lambda)^2|\mu|^2|z|^2}.$$

Using the Chern-Weil formula and integrating gives

$$\langle c_1(\mathcal{P}) \smile c_1(\mathcal{P}), [\mathrm{Pic}^0(T) \times T]\rangle = -2.$$

Since $\pi_{1*}\mathcal{P}^2 = 0$, the Grothendieck-Riemann-Roch theorem gives:

$$\begin{aligned}
\deg(c_1(R^1\pi_{1*}\mathcal{P}^2)) &= -\deg(\pi_{1*}\mathrm{ch}(\mathcal{P}^2)) \\
&= -\deg(\pi_{1*}[\frac{1}{2}c_1(\mathcal{P}^2)^2]) \\
&= -\deg(\pi_{1*}[2(c_1(\mathcal{P})^2)]) \\
&= 4.
\end{aligned}$$

By Lemma 2.2.16 the sheaf $R^1\pi_{1*}\mathcal{P}^2$ is supported at each of the 4 half-periods in $\mathrm{Pic}^0(T)$ and by the above has length 1 at each. $\boxed{5.3.18}$

An application of 5.3.17 then gives:

Corollary 5.3.19. *If for $1 \leq i \leq N_0$, $m_i = 1$ and $\mathrm{Supp}(Z) \cap \bar{\pi}^{-1}(y_i) = \emptyset$, then $\bar{\pi}^{-1}(y_1)$ is not contained in the support of $\bar{F}$.*

Lemma 5.3.20. *The line bundle $\mathcal{L}$ satisfies $-c_1(\mathcal{L})^2 = 2r$.*

Proof of 5.3.20: Let d be the degree of the graph γ of $\mathcal{L}$ and $\widetilde{H}$ be the pull-back to $\Sigma \times T$ of the divisor H' on $\mathbf{P}^1 \times T$ given by $H' = \tilde{q}^*H$. Taking intersections gives $4d = \gamma.\widetilde{H}$. Let x be a point in $\mathbf{P}^1 \times \mathbf{P}^1$ with $m = G._xH \neq 0$. Refering to the local resolutions given in Section 5.1, we see that if m is odd then x has one pre-image in Σ, contributing local intersection number m to $\gamma.\widetilde{H}$. If m is even then x has two pre-images in Σ, each of which contributes $m/2$ to $\gamma.\widetilde{H}$. Thus, $\gamma.\widetilde{H} = G.H = 4r$. Using 2.2.21 then gives the result. $\boxed{5.3.20}$

Corollary 5.3.21. *The push-down bundle $\tilde{\nu}_*\mathcal{L}$ satisfies $c_2(\tilde{\nu}_*\mathcal{L}) = r$.*

Proof of 5.3.21: Since $\tilde{\nu}$ is of degree 2, $c_2(\tilde{\nu}^*\tilde{\nu}_*\mathcal{L}) = 2c_2(\tilde{\nu}_*\mathcal{L})$. The result follows by applying the Chern character formula to the first column of the array in (5.3.10) and by (2.2.21). $\boxed{5.3.21}$

5.4. Construction of the strata

The results of Section 5.3 show that associated to each class $\mathcal{E}$ in $\mathcal{M}_k$ we have three items of data:

(a) Its graph G, which determines double covers $\nu : \Sigma \to G$ and $\bar{\nu} : \bar{X} \to X$, together with a pair of line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $\bar{X}$, whose graphs are the components in $\Sigma \times T$ of the pull-back of G.

(b) A torsion sheaf Q on X supported on an effective divisor F and clusters Z_1, Z_2 on X constrained by 5.3.11.

(c) An extension class in $\mathrm{Ext}^1(Q, \mathcal{F})$, where $\mathcal{F} \cong \bar{\nu}_*\mathcal{L}_1 \cong \bar{\nu}_*\mathcal{L}_2$ modulo the actions of $\mathrm{Aut}(\mathcal{F})$ and $\mathrm{Aut}(Q)$.

In principle we ought to able to reconstruct the bundle given this data. For $r \geq 1$, fixing a smooth $(r,1)$ curve on $\mathbf{P}^1 \times \mathbf{P}^1$ determines a (possibly singular) curve C and its normalisation Σ. From this we obtain a double cover $\bar{\nu} : \bar{X} \to X$, and a choice of line bundle $\mathcal{L}$ with appropriate graph produces $\mathcal{F}$. Next we have to choose Q (i.e., F and the Z_i) according to the relevant constraints, and finally we need an extension class in $\mathrm{Ext}^1(Q, \mathcal{F})$ which gives rise to a locally free sheaf.

Provided we parameterise the data in a sufficiently 'nice' way, one would expect to be able to construct substrata of the $\mathcal{M}_k^r$ which are complex manifolds. Unfortunately, in all but the simplest cases, the nature of the data does not lead to a very concrete description of the structure of the strata. However, we can give estimates of the dimensions of the strata, and determine to some extent the structure of the generic stratum. We follow the lines of Friedman, but the estimates arising from the choice of line bundle differ slightly and a little additional argument is necessary in order to identify the generic stratum.

Let us then consider how to construct families of classes in $\mathcal{M}_k^r$. First of all we wish to construct a family of double covers over the space of graphs. In order that the dimensions of the space of corresponding line bundles be constant, we need to ensure that the genus of the curves in the family remains constant. This we do by stratifying the space Rat_r of degree r rational maps $\mathbf{P}^1 \to \mathbf{P}^1$ by 'equisingular' curves. The approach uses a result of Zariski [48] and is simpler than the version given by Friedman.

Let h_i, $i = 1, \ldots, 4$ be the images in $\mathbf{P}^1$ of the half-periods of T under the hyperelliptic quotient. Let $H_i = \mathbf{P}^1 \times \{h_i\}$ the associated $(1,0)$ divisor on $\mathbf{P}^1 \times \mathbf{P}^1$. If G is a smooth $(r,1)$ divisor, for some $r \geq 1$, then $G.H_i = r$ for each $r = 1, \ldots, 4$. If G intersects H_i at M_i distinct points $x_1^{(i)}, \ldots, x_{M_i}^{(i)}$ with local intersection numbers $\alpha_1^{(i)}, \ldots, \alpha_{M_i}^{(i)}$ respectively, then for $i = 1, \ldots, 4$.

$$\sum_{j=1}^{M_i} \alpha_j^{(i)} = r,$$

and we obtain an associated partition $\underline{\alpha}^{(i)}$ of r. Denote the associated quadruple of partitions $(\underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(4)})$ by $\underline{\alpha}$, and for a fixed pair $(r, \underline{\alpha})$ let $V(r, \underline{\alpha})$ be the subvariety of $\mathbf{P}^{2r+1}$ consisting of smooth $(r,1)$ divisors which for $i = 1,\ldots,4$ intersect H_i in *some* set of distinct points $x_j^{(i)}, j = 1,\ldots,M_i$ with associated partition $\alpha^{(i)}$. Note that the integers $\alpha_j^{(i)}$ are just a re-ordering of the integers m_k of Section 5.1. We are therefore considering families of curves G with the same intersection behaviour with $H = \sum_{i=1}^4 H_i$, or from an alternative viewpoint, prescribing the nature of the singularities of the singular curve $G = H$ whilst allowing G, and therefore the precise position of the singularities, to vary.

A result of Zariski [48] shows that the projective tangent space to $V(r,\alpha)$ at a point G may be identified with a projective subspace of

$$\mathbf{P}(H^0(G;\mathcal{K}_G\otimes\mathcal{K}_Q^{-1})\oplus H^0(H;\mathcal{K}_H\otimes\mathcal{K}_Q^{-1})),$$

where we denote the quadric $\mathbf{P}^1\times\mathbf{P}^1$ by Q for simplicity. Zariski considers the singular curve $C = G + H$ on Q given by a non-trivial section σ_0 of $H^0(Q;\mathcal{O}_Q(G+H))$ and local defining equation $F(x,y) = 0$ in a neighbourhood of a point of intersection p with local intersection number α. Taking a smooth family of curves C_t in in the linear system $|\mathcal{O}_Q(G+H)|$, with $C_0 = C$, with each C_t defined locally by an equation $F(x,y,t) = 0$, he shows that the tangent curve γ at C given by

$$F_1(x,y) \stackrel{\text{def}}{=} \frac{\partial F}{\partial t}(x,y,0) = 0,$$

satisfies certain conditions. More precisely, the meromorphic 1-form ω on Q given by

$$\omega = \frac{F_1(x,y)}{\partial F/\partial y}dx,$$

restricts to a holomorphic 1-form on the two branches G and H of C near p. Hence γ defines a unique element of

$$\mathbf{P}(H^0(G;\mathcal{K}_G\otimes\mathcal{K}_Q^{-1})\oplus H^0(H;\mathcal{K}_H\otimes\mathcal{K}_Q^{-1})),$$

independent of the choice of lifted defining function $F(x,y,t)$.

In our case, for a fixed $i\in\{1,\ldots,4\}$ we may take a coordinate system centred at p such that the line H_i is given by $y = 0$. The equation of G near p may be written locally as $f(x,y) = y - x^\alpha = 0$, for some local coordinate x on $\mathbf{P}^1$. We consider only variations of the curve G; i.e, variations of f of the form

$$f(x,y,t) = y - (x + t\xi(x) + t^2\eta(x) + \ldots)^\alpha.$$

Thus, in the notation of Zariski we have

$$F(x,y,t) = y(y - (x + t\xi(x) + \ldots)^\alpha).$$

The meromorphic 1-form ω is then given by

$$\omega = \frac{-\alpha y x^{\alpha-1}\xi(x)}{(y - x^\alpha) + y}dx.$$

On the curve G, given locally by $x^{\alpha} = y$ with local coordinate x, the 1-form ω restricts to

$$\omega_G = -\alpha x^{\alpha-1}\xi(x)dx,$$

and the restriction of ω to H_i vanishes. Hence the image of the affine tangent space to $V(r,\underline{\alpha})$ at G is a vector subspace W_G of $H^0(G;\mathcal{K}_G\otimes\mathcal{K}_Q^{-1})\oplus H^0(H;\mathcal{K}_H\otimes\mathcal{K}_Q^{-1})$, consisting of forms (α_G,α_H) where $\alpha_H = 0$ and α_G is divisible by $x^{\alpha_j^{(i)}-1}$ at each point of intersection $x_j^{(i)}$. Since the kernel of this map is 1-dimensional, generated by the section σ_0, we have $\dim V(r,\alpha) = \dim W_G$. This fact allows us to estimate the dimension of $V(r,\alpha)$.

The curve G is isomorphic to $\mathbf{P}^1$ so $\deg(\mathcal{K}_G\otimes\mathcal{K}_Q^{-1}|_G) = 2r$ giving $h^0(G;\mathcal{K}_G\otimes\mathcal{K}_Q^{-1}|_G) = 2r+1$. We are interested in the subspace of global sections of $\mathcal{K}_G\otimes\mathcal{K}_Q^{-1}|_G$ which vanish to the prescribed degrees $\beta_j^{(i)} = \alpha_j^{(i)} - 1$ at the points $x_j^{(i)}\in H_i$; i.e, those sections of $H^0(G;\mathcal{O}_G(2r))$ which map to 0 under the restriction map ρ in the exact sequence

$$0\to H^0(\mathcal{O}_G(2r-\sum_{i,j}\beta_j^{(i)}x_j^{(i)}))\to H^0(\mathcal{O}_G(2r))\xrightarrow{\rho} H^0(\mathcal{O}_Y),$$

induced by the exact sequence of sheaves

$$0\to\mathcal{O}_G(2r)\otimes\mathcal{O}_G(-\sum_{i,j}\beta_j^{(i)}x_j^{(i)})\to\mathcal{O}_G(2r)\to\mathcal{O}_Y\to 0,$$

where $\mathcal{O}_Y = \mathcal{O}_G/\mathcal{O}_G(-\sum_{i,j}\beta_j^{(i)}x_j^{(i)})$ is the structure sheaf of the 0-dimensional subscheme $Y = \sum_{i,j}\beta_j^{(i)}x_j^{(i)}$ of H. Hence the dimension of the $V(r,\alpha)$ is bounded above by $h^0(\mathcal{O}_G(2r-\sum_{i,j}\beta_j^{(i)}x_j^{(i)}))$ and we have the following:

Lemma 5.4.1. *The dimension of $V(r,\underline{\alpha})$ satisfies*

$$\dim V(r,\underline{\alpha}) \le 2r+1-\sum_{i,j}\beta_j^{(i)}.$$

Remark 5.4.2. The lemma shows that $V(r,\underline{\alpha})$ is empty if

$$\sum_{i,j}(\alpha_j^{(i)}-1) > 2r+1.$$

Next we show that for a fixed pair $(r,\underline{\alpha})$ the space of compatible line bundles has constant dimension over our family of curves $V(r,\alpha)$.

Lemma 5.4.3. *If $\mathcal{L},\mathcal{L}'\in \mathrm{Pic}^0(\overline{X},\Sigma)$ both have graph γ then they satisfy $\det\tilde{\nu}_*\mathcal{L}\cong\det\tilde{\nu}_*\mathcal{L}'\cong\mathcal{O}_X(-n)$, for some $n\in\mathbf{Z}$, if and only if $\mathcal{L}\cong\mathcal{L}'\otimes\bar{\pi}^*\mathcal{U}$ for some $\mathcal{U}\in\mathrm{Pic}^0(\Sigma)$.*

Proof of 5.4.3: If $\mathcal{L},\mathcal{L}'$ have the same graph, then $\mathcal{L}\cong\mathcal{L}'\otimes\pi^*\mathcal{U}$ for some $\mathcal{U}\in\mathrm{Pic}(\Sigma)$. Using U to denote the divisor class of $\mathcal{U}$ on Σ, we have by

Lemma 5.1.9:

$$\begin{aligned}\det \tilde{\nu}^*\tilde{\nu}_*\mathcal{L} &\cong \mathcal{L}\otimes\iota^*\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-R)\\ &\cong \mathcal{L}'\otimes\iota^*\mathcal{L}'\otimes\mathcal{O}_{\tilde{X}}(-R)\otimes\tilde{\pi}^*(\mathcal{U}\otimes\iota^*\mathcal{U})\\ &\cong \det\tilde{\nu}^*\tilde{\nu}_*\mathcal{L}'\otimes\pi^*(\mathcal{U}\otimes\iota^*\mathcal{U})\end{aligned}$$

Hence $U + \iota^*U = \nu^*\nu_*U \sim 0$ giving $\deg U = 0$. Conversely, if $\deg U = 0$ then $U = \sum a_j P_j$, where each P_j is a point of Σ and the a_j are integers with $\sum a_j = 0$. This gives

$$U + \iota^*U = \sum a_j(P_j + \iota^*P_j).$$

Since Σ is hyperelliptic, for each j we have [32, p.342]

$$P_j + \iota^*P_j \sim P_1 + \iota^*P_1.$$

Hence

$$U + \iota^*U \sim (\textstyle\sum a_j)(P_1 + \iota^*P_1) \sim 0.$$

Therefore $\deg(\nu_*U) = 0$ giving $\deg(U) = 0$. $\boxed{5.4.3}$

Lemma 5.4.4. *Suppose $\mathcal{L}\in \mathrm{Pic}^0(\tilde{X},\Sigma)$ satisfies $\det\tilde{\nu}_*\mathcal{L}\cong\mathcal{O}_X(-n)$, for some $n\in\mathbb{Z}$. If $\mathcal{U}\in\mathrm{Pic}^0(\Sigma)$ and $\tilde{\nu}_*(\mathcal{L}\otimes\tilde{\pi}^*\mathcal{U})\cong\tilde{\nu}_*\mathcal{L}$, then $\mathcal{U}\cong\mathcal{O}_\Sigma$.*

Proof of 5.4.4: The bundle $\mathcal{E}\cong\nu_*\mathcal{L}$ has maximal destabilising line bundles $\mathcal{L}(-R)$ and $\mathcal{L}(-R)\otimes\pi^*\mathcal{U}$, both with graph γ. By (2.4.2), there is only one such maximal destabilising line bundle up to isomorphism. $\boxed{5.4.4}$

Lemmas 5.4.3 and 5.4.4 show that if we fix a line bundle $\mathcal{L}$ such that $\det\tilde{\nu}_*\mathcal{L}\cong\mathcal{O}_X(-n)$, then the space of isomorphism classes of line bundles $\mathcal{L}'$ such that $\tilde{\nu}_*\mathcal{L}'$ has the same determinant is an "affine space"

$$\mathbf{T} = \{\mathcal{L}\otimes\tilde{\pi}^*\mathcal{U} : \mathcal{U}\in\mathrm{Pic}^0(\Sigma)\}.$$

Hence by 5.4.1 and 5.1.5, we see that the dimension

$$\dim V(r,\alpha) + g(\Sigma)$$

of the space of moduli of pairs $(G,\mathcal{L})$ is bounded above by

$$\begin{aligned}&2r + 1 - \sum_{i,j}(\alpha_j^{(i)} - 1) + 2r - 1 - \sum_{i,j}[\alpha_j^{(i)}/2]\\ &= 4r - \sum_{i=1}^{N}(m_i - 1) - \sum_{i=1}^{N} n_i.\end{aligned}$$

We now fix α and consider only bundles whose graph lies in $V(r,\alpha)$. Having fixed the graph G of a map in $V(r,\alpha)$, and hence the corresponding curves C and Σ, we choose $\mathcal{L}$. In then selecting Q, we are free to choose those fibres in the support of F which are not constrained to coincide with fibres of X

where $\nu_* \mathcal{L}$ is an extension of half-periods, i.e., those fibres which correspond to singularities of C. An upper bound for the number of moduli for Q and the resulting group $\mathrm{Ext}^1(Q, \mathcal{F})$ can obtained using the results of Friedman with little modification. We therefore state the results and refer the reader to [24] for proofs.

Proposition 5.4.5 (Friedman, Prop.6.7). *The number of moduli for Q is bounded above by $h^1(\mathcal{O}_F) + \frac{1}{2}l(Z)$.*

Proposition 5.4.6 (Friedman). *The dimension of $\mathrm{Ext}^1(Q, \mathcal{F})$ is bounded above by $(\sum_{i=1}^{N} m_i) - N_0 + l(Z)$.*

Lemma 5.4.7. *The dimension of $\mathrm{Aut}(Q)$ is bounded below by $h^0(\mathcal{O}_F)$.*

Proof of 5.4.7: The sheaf $\mathcal{O}_F^*$ acts by automorphisms on Q. $\boxed{5.4.7}$

Summing the bounds (5.4.1), (5.4.5) and (5.4.6) for the moduli and subtracting that for the action of $\mathrm{Aut}(Q)$, we obtain an upper bound $d(\alpha)$ for the number of moduli in our family:

$$\begin{aligned} d(\underline{\alpha}) &= 4r - (\sum_{i=1}^{N} n_i) + N - N_0 + \frac{3}{2}l(Z) + h^1(\mathcal{O}_F) - h^0(\mathcal{O}_F) \\ &= 4r + \frac{3}{2}l(Z) - \chi(\mathcal{O}_F) - \sum_{i=1}^{N_0} n_i - \sum_{i=N_0+1}^{N} (n_i - 1). \qquad \textbf{5.4.8.} \end{aligned}$$

Taking the Euler characteristic of the structure sequence of F,

$$0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0,$$

gives $\chi(\mathcal{O}_F) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-F)) = 0$, and the bound becomes

5.4.9. $$d(\underline{\alpha}) = 4r + \frac{3}{2}l(Z) - \sum_{i=1}^{N_0} n_i - \sum_{i=N_0+1}^{N} (n_i - 1).$$

The dimension of $\mathcal{M}_k$ is $4k = 4r + 2l(Z)$. Comparing this with 5.4.9, we see that

$$4k - d(\alpha) = \sum_{i=1}^{N_0} n_i + \sum_{i=N_0+1}^{N} (n_i - 1) + \frac{1}{2}l(Z) \geq 0.$$

This difference vanishes if and only if we have

(1) $l(Z) = 0$;

(2) $n_i = 0$, for $i = 1, \ldots, N_0$;

(3) $n_i = 1$, for $i = N_0 + 1, \ldots, N$.

Condition (1) says there are no fibres of type (3), and conditions (2) and (3) tell us that the graph intersects the divisor H in points with local intersection number at most 2. The associated double cover C therefore has singularities which are at worst nodes. These singularities correspond to the collection of points $\{x_j : j = N_0 + 1, \ldots, N\}$ of $\mathbf{P}^1$ which must lie outside the branch locus of $\nu : \Sigma \to \mathbf{P}^1$. By (5.3.15) and (5.3.17) we see that

$$F = \sum_{j=N_0+1}^{N} \gamma_j F_j,$$

where $F_j = \pi^{-1}(x_j)$ and $\gamma_j = 0$ or 1. Given the pair $(G, \mathcal{L})$, the sheaf Q is then completely determined by the condition (5.3.11), namely we must have

5.4.10. $$\tilde{Q} \cong \mathcal{O}_{\tilde{F}} \otimes \mathcal{L}^{-1},$$

where $\tilde{F} = \nu^* F$. Since the fibres F_i of X on which Q is supported lie outwith the ramification divisor of $\tilde{\nu}$, for each $i = N_0 + 1, \ldots, N$, the divisor $\tilde{\nu}^*(F_i)$ consists of two components $F_i^{(1)}$ and $F_i^{(2)}$ interchanged by the involution ι. Each component has a neighbourhood on which the restriction of $\tilde{\nu}$ defines an isomorphism with a neighbourhood of F_i. Thus (5.4.10) determines Q and there is no contribution to the number of moduli for Q in this case. Since $l(Z) = 0$, the dimension of the strata corresponding to $\underline{\alpha}$ is bounded above by

$$\begin{aligned} d(\underline{\alpha}) &= 2k + 1 - \sum_{k=1}^{N}(m_k - 1) + 2k - 1 - \sum_{i=1}^{N} n_i + \sum_{i=1}^{N} m_i - N_0 - h^0(\mathcal{O}_F) \\ &= 4k - \sum_{i=1}^{N_0} n_i - \sum_{i=N_0+1}^{N} (n_i - 1) - h^0(\mathcal{O}_F). \end{aligned}$$

This is equal to $4k$ if and only if $F = \emptyset$. The only contribution to the moduli then comes from the pairs $(G, \mathcal{L})$, but from (5.4.5) we see this is equal to $4k$ if and only if $N = N_0$ and $m_i = 1$ for $i = 1, \ldots, N$. Hence the generic stratum in $\mathcal{M}_k$ corresponds to taking $m_i = k$ and $\alpha_j^{(i)} = 1$ for $j = 1, \ldots, k$ and $i = 1, \ldots, 4$. We then obtain:

Theorem 5.4.11. *For $k > 0$, the generic stratum in $\mathcal{M}_k$ consists of isomorphism classes of bundles $\mathcal{E}$ whose graphs are transverse to the divisor H. For each such bundle there is a double cover $\tilde{\nu} : \tilde{X} \to X$ ramified at a divisor R on $\tilde{X}$ and two line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $\tilde{X}$, uniquely determined up to isomorphism, such that $\mathcal{E} \cong \tilde{\nu}_*(\mathcal{L}_i(R))$ for $i = 1, 2$.*

Let the integer $k \geq 1$ be fixed. We make the following definitions.

Definition 5.4.12. Let Rat_k^* denote the open subvariety of $\mathbf{P}H^0(\mathcal{O}(k,1))$ consisting of smooth $(k,1)$ divisors G on $\mathbf{P}^1 \times \mathbf{P}^1$ which satisfy

$$\max\{G._x H : x \in H\} = 1,$$

i.e., graphs of degree k rational maps $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$ for which the image in the target space $\mathbf{P}^1$ of the half-periods of T are regular points.

Definition 5.4.13. Let $\mathcal{M}_k^*$ denote the pre-image of Rat_k^* under the divisor map

$$D : \mathcal{M}_k \to \mathbf{P}H^0(\mathcal{O}(k,1)).$$

We have the following quite general result:

Lemma 5.4.14. *Let $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$ be a degree r map, $(r \geq 1)$. For any $n \in \mathbf{Z}$ there is a bundle $\mathcal{F} \in SL_0(X, \mathbf{P}^1)$ with $D(\mathcal{F}) = G_\alpha$, $\det \mathcal{F} \cong \mathcal{O}_X(n)$ and $c_2(\mathcal{F}) = r$.*

Proof of 5.4.14: Let $\tilde{\nu} : \tilde{X} \to X$ be the double cover corresponding to α. Choose a reduction $\gamma + \iota^*\gamma$ of the divisor $G_{\alpha\circ\nu}$ on $\Sigma \times T$. Let $\mathcal{L}$ be a line bundle in $\mathrm{Pic}^0(\tilde{X}, \Sigma)$ with graph γ. The bundle $\mathcal{L} \otimes \iota^*\mathcal{L}$ is ι-invariant and restricts to the trivial bundle on all fibres of $\tilde{\pi}$. Consequently $\mathcal{L} \otimes \iota^*\mathcal{L} \cong \tilde{\nu}^*\pi^*\mathcal{O}_{\mathbf{P}^1}(m)$ for some $m \in \mathbf{Z}$, and

$$\tilde{\nu}^* \det \tilde{\nu}_*\mathcal{L} \cong (\tilde{\nu}^*\pi^*\mathcal{O}_{\mathbf{P}^1}(m)) \otimes \mathcal{O}_{\tilde{X}}(-R) \cong \tilde{\nu}^*\pi^*\mathcal{O}_{\mathbf{P}^1}(m-d),$$

where $d = \deg \mathcal{O}_X(D)$. Suppose rf represents the divisor class of $\det \tilde{\nu}_*\mathcal{L}$, where f is a fibre of π; then $\tilde{\nu}^* rf = \tilde{\nu}^*(m-d)f$ and since $\tilde{\nu}$ is of degree 2,

$$\tilde{\nu}_*\tilde{\nu}^* rf = 2rf = 2(m-d)f.$$

Hence, since all divisors on X are linearly equivalent to multiples of f, we obtain $r = m - d$. Now let $\tilde{f}$ be a fibre of $\tilde{\pi} : \tilde{X} \to \Sigma$ and let $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_{\tilde{X}}((d - m + n)\tilde{f})$. Then

$$\tilde{\nu}^* \det \tilde{\nu}_*\mathcal{L}' \cong (\tilde{\nu}^* \det \nu_*\mathcal{L}) \otimes \mathcal{O}_{\tilde{X}}((d - m + n)(\tilde{f} + \iota^*\tilde{f})),$$

giving

$$\det \tilde{\nu}_*\mathcal{L}' \cong \det \tilde{\nu}_*\mathcal{L} \otimes \mathcal{O}_X(d - m + n) \cong \mathcal{O}_X(n).$$

The bundle $\mathcal{F} = \tilde{\nu}_*\mathcal{L}'$ has $c_2(\mathcal{F}) = r$ by (5.3.20) and so fulfills the properties required by the lemma. $\boxed{5.4.14}$

For each $\varphi \in \mathrm{Rat}_k^*$, let $G_\varphi \subset \mathbf{P}^1 \times \mathbf{P}^1$ be its graph and denote by Σ_φ the corresponding non-singular double cover. The structure of the generic stratum $\mathcal{M}_k^*$ is summarised in the the following

Corollary 5.4.15. *The map $D : \mathcal{M}_k^* \to \mathrm{Rat}_k^*$ is surjective, with fibres diffeomorphic to $(S^1 \times S^1)^{2k-1}$.*

Proof of 5.4.15: Surjectivity follows from (5.4.14). By (5.4.3) and (5.4.4) the fibre of D over a point $\varphi \in \mathrm{Rat}_k^*$ is diffeomorphic to $\mathrm{Pic}^0(\Sigma_\varphi)$. Since the genus of Σ_φ is given by $2k-1$, the result then follows. $\boxed{5.4.15}$

5.5. Charge 1 instantons: the upper stratum

The case $k = 1$ is exceptional in the sense that the graphs of *all* degree 1 maps $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$ are transverse to the divisor H and hence $\mathcal{M}_1^1 = \mathcal{M}_1^*$. The corresponding double covers $\Sigma \subset \mathbf{P}^1 \times T$ are all isomorphic to T via the projection $\pi_2 : \mathbf{P}^1 \times T \to T$. This fact allows us to work with a fixed double cover of $\mathbf{P}^1$ and give a global description of the generic stratum. Indeed there is a special divisor Σ_0 on $\mathbf{P}^1 \times T$, namely the graph of the hyperelliptic quotient $q : T \to \mathbf{P}^1$, and $\pi_1 : \Sigma_0 \to \mathbf{P}^1$ is the double cover resulting from the diagonal $\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$, i.e, the graph of the identity map on $\mathbf{P}^1$. If now α is any automorphism of $\mathbf{P}^1$, the corresponding curve Σ is the graph of the map $\alpha^{-1} \circ q : T \to \mathbf{P}^1$. Thus we can equally as well use the fixed double cover $\bar{q} : \bar{X} \to X$ obtained from the map $q : T \to \mathbf{P}^1$, and vary the degree 2 morphism by composing with automorphisms of X. As we have seen, every $\mathrm{SL}(2,\mathbf{C})$ bundle $\mathcal{E}$ on X with $D(\mathcal{E}) = \Delta$ is isomorphic to $\bar{q}_*\mathcal{L}$ for a unique class of bundle $\mathcal{L} \in \mathrm{Pic}^0(\bar{X}, T)$ with graph the identity map of T. If $D(\mathcal{E}) = D(\mathcal{E}') = \Delta$, the corresponding line bundles $\mathcal{L}, \mathcal{L}'$ satisfy $\mathcal{L} \cong \mathcal{L}' \otimes \pi^*\mathcal{U}$ for some $\mathcal{U} \in \mathrm{Pic}^0(T) \cong T$. Given an automorphism α of $\mathbf{P}^1$, let $\tilde{\alpha} \in \mathrm{Aut}_0(X)$ be a lift of α, then $\tilde{\alpha}^*\bar{q}_*\mathcal{L}$ has graph α. Conversely, if $\mathcal{E}$ has graph G_α for some automorphism α of $\mathbf{P}^1$ and $\tilde{\alpha}$ is a lift of α to $\mathrm{Aut}_0(X)$, then $(\tilde{\alpha}^{-1})^*\mathcal{E}$ has divisor Δ and so is of the form $\bar{q}_*\mathcal{L}$ for some line bundle $\mathcal{L}$ on $\bar{X}$.

Definition 5.5.1. Let $\mathbf{T}$ denote the subset of $\mathrm{Pic}^0(\bar{X}, T)$ consisting of bundles $\mathcal{L}$ with graph the identity map of T and such that $\det \bar{q}_*\mathcal{L} \cong \mathcal{O}_X$.

By (5.4.3), $\mathbf{T}$ has a natural free and transitive $\mathrm{Pic}^0(T)$-action. The above discussion proves:

Proposition 5.5.2. *If $\mathcal{E} \in \mathcal{M}_1^1$ has graph G_α for some automorphism α of $\mathbf{P}^1$, then $\mathcal{E}$ is isomorphic to $\tilde{\alpha}^*\bar{q}_*\mathcal{L}$ for some $\mathcal{L} \in \mathbf{T}$ and lift $\tilde{\alpha}$ of α to* $\mathrm{Aut}_0(X)$.

Hence we have a surjective map:

$$\begin{aligned} \Theta : \mathrm{Aut}_0(X) \times \mathbf{T} &\longrightarrow \mathcal{M}_1^1 \\ (\varphi, \mathcal{L}) &\mapsto \varphi^*\bar{q}_*\mathcal{L}, \end{aligned} \qquad 5.5.3.$$

such that $D(\Theta(\varphi, \mathcal{L})) = G_\alpha$, where α is the automorphism of $\mathbf{P}^1$ induced by φ.

It will be useful to know how the elements of $\mathbf{T}$ behave under the action of T on $\bar{X}$. We shall therefore give a more concrete description of the space $\mathbf{T}$. First note that $\bar{X}$ is the quotient by $\mathbf{Z}$ of the $\mathbf{C}^*$-bundle associated to the degree 2 line bundle $\mathcal{V} = q^*\mathcal{O}_{\mathbf{P}^1}(-1)$. Following the construction detailed in [30, p.307], the line bundle $\mathcal{V}$ is isomorphic to a bundle over $T = \mathbf{C}^*/\langle\lambda\rangle$ given by taking the quotient of $\mathbf{C}^* \times \mathbf{C}$ by the $\mathbf{Z}$-action generated by $(z, w) \mapsto (\lambda z, e(z)w)$, where $e : \mathbf{C}^* \to \mathbf{C}^*$ is a holomorphic map of degree 2 determined by the isomorphism class of $\mathcal{V}$.

Hence $\bar{X}$ is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the $\mathbf{Z} \times \mathbf{Z}$ action with generators:

$$\begin{aligned} (1,0).(z, w) &= (\lambda z, e(z)w); \\ (0,1).(z, w) &= (z, \lambda w), \end{aligned}$$

with the projection $\bar{\pi} : \bar{X} \to T$ given by $\langle(z, w)\rangle \mapsto \langle z\rangle$. This action extends to one on $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$ with generators:

$$\begin{aligned} (1,0).(z, w, \xi) &= (\lambda z, e(z)w, w\xi); \\ (0,1).(z, w, \xi) &= (z, \lambda w, z\xi). \end{aligned} \qquad 5.5.4.$$

The resulting quotient is a line bundle $\mathcal{L}_0$ over $\overline{X}$. By construction, $\mathcal{L}_0|_{\bar{\pi}^{-1}(t)} \cong \mathcal{V}_t$, where $\mathcal{V}_t$ is the bundle defined in (2.1), and so $\mathcal{L}_0$ has the identity map of T as its graph. Let R_μ denote the right action of $\mu \in T$ on $\overline{X}$ (or on X). The above construction leads directly to:

Lemma 5.5.5. *If $\mathcal{L} \in \mathbf{T}$, then $R_\mu^*\mathcal{L} \cong \mathcal{L} \otimes \bar{\pi}^*\mathcal{V}_\mu$.*

Proof of 5.5.5: There is a line bundle $\mathcal{U}$ on T such that $\mathcal{L} \cong \mathcal{L}_0 \otimes \bar{\pi}^*\mathcal{U}$. Now $R_\mu^*\mathcal{L} \cong R_\mu^*\mathcal{L}_0 \otimes \bar{\pi}^*\mathcal{U}$, but from (5.5.4) we see that $R_\mu^*\mathcal{L}$ is the bundle given by:

$$(1,0).(z,w,\xi) = (\lambda z, e(z)w, \mu w\xi);$$
$$(0,1).(z,w,\xi) = (z, \lambda w, z\xi),$$

from which we deduce that $R_\mu^*\mathcal{L} \cong \mathcal{L} \otimes W$, where W is the bundle with defining generators:

$$(1,0).(z,w,\xi) = (\lambda z, e(z)w, \mu\xi);$$
$$(0,1).(z,w,\xi) = (z, \lambda w, \xi).$$

But this is clearly isomorphic to $\bar{\pi}^*\mathcal{V}_\mu$, as required. $\boxed{5.5.5}$

This allows us to prove:

Lemma 5.5.6. *Let $\Theta : \mathrm{Aut}_0(X) \times \mathbf{T} \to \mathcal{M}_1^1$ be the map given by (5.5.3), then $\Theta(\varphi, \mathcal{L}) \cong \Theta(\varphi', \mathcal{L}')$ if and only if $\varphi' = R_\mu \circ \varphi$ and $\mathcal{L}' \cong \mathcal{L} \otimes \pi^*\mathcal{V}_\mu$ for some $\mu \in T$.*

Proof of 5.5.6: If $\Theta(\varphi, \mathcal{L}) \cong \Theta(\varphi', \mathcal{L}')$, i.e., $\varphi^*\bar{q}_*\mathcal{L} \cong (\varphi')^*\bar{q}_*\mathcal{L}'$, then since the respective graphs must be identical, the automorphisms φ and φ' induce the same automorphism of $\mathbf{P}^1$. Therefore $\varphi' = R_\mu\varphi$ for some $\mu \in T$. Consequently, $\bar{q}_*\mathcal{L} \cong R_\mu^*\bar{q}_*\mathcal{L}'$. By (5.5.5) and (5.4.4), we have $\mathcal{L} \cong \mathcal{L}' \otimes \bar{\pi}^*\mathcal{V}_\mu$ for a unique $\mu \in T$, giving the result. $\boxed{5.5.6}$

Proposition 5.5.7. *The moduli space $\mathcal{M}_1^1$ is diffeomorphic to* $\mathrm{Aut}_0(X)$.

Proof of 5.5.7: Defining an action of T on $\mathrm{Aut}_0(X) \times \mathbf{T}$ by

$$\mu.(\varphi, \mathcal{L}) = (\varphi \circ R_\mu^{-1}, \mathcal{L} \otimes \pi^*\mathcal{V}_\mu),$$

for $\mu \in T$ and quotienting out by this action gives by (5.5.6) a bijective holomorphic map

$$\mathrm{Aut}_0(X) \times_T \mathbf{T} \longrightarrow \mathcal{M}_1^1.$$

Since the action of T on $\mathbf{T}$ is free and transitive, this is a holomorphic bijection

$$\bar{\Theta} : \mathrm{Aut}_0(X) \longrightarrow \mathcal{M}_1^1.$$

Hence Θ is a diffeomorphism. $\boxed{5.5.7}$

Recall we have a natural action of $X \hookrightarrow \mathrm{Aut}_0(X)$ on $\mathcal{M}_1^1$. There is a natural free left action of X on $\mathrm{Aut}_0(X) \times \mathbf{T}$ given by

$$\psi.(\varphi, \mathcal{L}) = (\varphi \circ \psi^{-1}, \mathcal{L}).$$

This action commutes with the action of T. A simple computation shows that

$$\Theta(\psi.(\varphi, \mathcal{L})) = (\psi^{-1})^{*}\Theta(\varphi, \mathcal{L}),$$

and so the map Θ is equivariant with respect to the natural action of X on $\mathcal{M}_1^1$, giving:

Proposition 5.5.8. *The natural action of X on $\mathcal{M}_1^1$ is free and the diffeomorphism $\Theta : \mathrm{Aut}_0(X) \to \mathcal{M}_1^1$ is X-equivariant.*

Combining this with (3.7.5) gives:

Theorem 5.5.9. *The natural left action of $S^1 \times S^3$ on $\mathcal{M}_1$ is free.*

The subgroup T of $\mathrm{Aut}_0(X)$ consisting of the image in $\mathrm{GL}(2, \mathbf{C})/\mathbf{Z}$ of the non-zero diagonal matrices acts holomorphically on $\mathcal{M}_1$. The orientation-preserving diffeomorphism on X defined on the quaternions $\mathbf{H}^*$ by $z \mapsto z^{-1}$ carries the action of T to an action of a subgroup of $S^1 \times S^3$. Hence the holomorphic action of T on $\mathcal{M}_1$ is free and leads to:

Theorem 5.5.10. *The space $D : \mathcal{M}_1 \to \mathbf{P}^3 \setminus (\mathbf{P}^1 \times I)$ is a principal elliptic fibration.*

We defer the proof of (5.5.10) until the next chapter, where it will be proved in greater generality for the product of the circle with certain Lens spaces.

Chapter 6

Stable $\mathrm{SL}(2,\mathbb{C})$ bundles over $S^1\times$Lens space

Having given a description of the moduli space $\mathcal{M}_1$ associated to $S^1 \times S^3$, we can now do the same for $S^1 \times L_p$, where L_p is a Lens space. Indeed we shall see that $S^1 \times L_p$ can be given the structure of a principal elliptic fibration over $\mathbf{P}^1$. The results of Chapters 2, 3 and 5 apply more generally to such spaces. In particular, the fibrations are locally trivial so all fibres are analytically isomorphic and there are no multiple or singular fibres. The canonical bundle of the total space is then just the pull-back of the canonical bundle of $\mathbf{P}^1$, and all formulae relating maximal destabilising line bundles etc. go through unchanged. The only significant difference is the appearance of torsion in integral cohomology which leads to extra structure in the moduli space.

6.1. Lens spaces

Recall that the Lens space $L_p = L(1,p)$ may be defined to be the quotient of $S^3 = \{(z_0, z_1) \in \mathbf{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$ by the $\mathbf{Z}_p$-action generated by:

6.1.1. $$(z_0, z_1) \mapsto e^{2\pi i/p}(z_0, z_1).$$

The space $S^1 \times L_p$ is then diffeomorphic to the complex manifold X_p given by the quotient of $\mathbf{C}^2 \setminus \{0\}$ by the $\mathbf{Z} \times \mathbf{Z}_p$-action with generators defined by (6.1.1) and

$$(z_0, z_1) \mapsto \lambda(z_0, z_1),$$

where $\lambda \in \mathbf{R}$ with $|\lambda| > 1$. The $\mathbf{Z} \times \mathbf{Z}_p$-action extends in an obvious way to a free holomorphic $\mathbf{C}^*$-action defining a principal elliptic fibration

$$\pi : X_p \to \mathbf{P}^1,$$

with fibres $\mathbf{C}^*/\mathbf{Z} \times \mathbf{Z}_p \cong \mathbf{C}^*/\{\lambda, e^{2\pi i/p}\} \cong T'$, where T' is the elliptic curve $\mathbf{C}^*/\{\lambda^p\}$. In the language of Kodaira X_p is a "secondary Hopf surface".

Alternatively, we may think of X_p as the quotient of the principal $\mathbf{C}^*$-bundle associated to the line bundle $\mathcal{O}_{\mathbf{P}^1}(-p)$ by the action of $\mathbf{Z}$ generated by scalar multiplication by λ^p in the fibres. The Hopf surface X of Chapters 3, 4 and 5 is isomorphic to X_1 and the degree p cover $X \to X_p$ is the natural one induced by tensor multiplication

$$\mathcal{O}_{\mathbf{P}^1}(-1) \xrightarrow{\otimes p} \mathcal{O}_{\mathbf{P}^1}(-p).$$

The group $\mathrm{GL}(2,\mathbf{C})$ acts on the left of X_p by automorphisms, covering the action of $\mathrm{PGL}(2,\mathbf{C})$ on $\mathbf{P}^1$. The subgroup S generated by $\lambda.\mathrm{id}$ and $e^{2\pi i/p}.\mathrm{id}$ acts trivially, giving an identification

$$\mathrm{Aut}_0(X_p) \cong \mathrm{GL}(2,\mathbf{C})/S.$$

Note that the projection $\mathrm{GL}(2,\mathbf{C}) \to \mathrm{PGL}(2,\mathbf{C})$ gives $\mathrm{Aut}_0(X_p)$ the structure of a principal T'-bundle over $\mathrm{PGL}(2,\mathbf{C})$.

The integral cohomology of X_p is given by

$$H^*(X_p;\mathbf{Z}_p) = (\mathbf{Z},\mathbf{Z},\mathbf{Z}_p,\mathbf{Z}\times\mathbf{Z}_p,\mathbf{Z}).$$

Consequently $h^{0,2}(X_p) = 0$ and by Noether's theorem $h^{0,1}(X_p) = 1$. The long exact sequence in cohomology induced by

$$0 \to \mathbf{Z} \to \mathcal{O}_{X_p} \to \mathcal{O}^*_{X_p} \to 0$$

gives

$$\mathrm{Pic}(X_p) = H^1(X_p,\mathcal{O}^*_{X_p}) \cong \mathbf{C}^* \times \mathbf{Z}_p.$$

An explicit representative $\mathcal{V}(\mu,r)$ for the element defined by $(\mu,r) \in \mathbf{C}^* \times \mathbf{Z}_p$ is given by the line bundle constructed as the quotient of $(\mathbf{C}^2 \setminus \{0\}) \times \mathbf{C}$ by the action of $\mathbf{Z} \times \mathbf{Z}_p$ with generators:

$$\begin{aligned}(1,0).(z,\xi) &= (\lambda z,\mu\xi)\\ (0,1).(z,\xi) &= (e^{2\pi i/p}z, e^{2\pi r i/p}\xi).\end{aligned}$$

The function on $\mathbf{C}^2 \setminus \{0\}$ defined by $(z_0,z_1) \mapsto z_0$ defines a section of $\mathcal{V}(\lambda,1)$ with divisor $F = \pi^{-1}([0,1])$. Denoting $\pi^*\mathcal{O}_{\mathbf{P}^1}(n)$ by $\mathcal{O}_{X_p}(n)$, we then have $\mathcal{O}_{X_p}(F) \cong \mathcal{O}_{X_p}(1)$ and $\mathcal{K}_{X_p} \cong \mathcal{O}_X(-2)$.

Since $H^2(X_p;\mathbf{Z})$ is pure torsion, any line bundle on X_p restricts to a degree 0 line bundle on fibres of π, so the exact sequence (2.2.10) becomes:

6.1.2.
$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Pic}(\mathbf{P}^1) & \to & \mathrm{Pic}(X_p) & \to & \mathrm{Map}(\mathbf{P}^1,T') & \to & 1\\ & & \| & & \| & & \| & & \\ 0 & \to & \mathbf{Z} & \xrightarrow{\pi^*} & \mathbf{C}^*\times\mathbf{Z}_p & \xrightarrow{\rho} & T' & \to & 1,\end{array}$$

where $\pi^*(n) = (\lambda^n, n)$. We can determine the map ρ as follows. Recall [30, p.308] that if we regard T' as the quotient of $\mathbf{C}$ by the lattice with generators $\lambda_1 = p\log\lambda$ and $\lambda_2 = 2\pi i$, then any holomorphic line bundle $\mathcal{V}$ on T' is isomorphic to the quotient of $\mathbf{C}\times\mathbf{C}$ by an equivalence relation generated by

$$(z,\xi) \sim (z+\lambda_1, e_1(z)\xi) \sim (z+\lambda_2, e_2(z)\xi),$$

where $e_1, e_2 : \mathbf{C} \to \mathbf{C}^*$ are holomorphic maps satisfying certain compatibility conditions. Up to isomorphism it is always possible to choose $e_2 \equiv 1$. The action of the lattice subgroup generated by λ_2 is then trivial on the fibres and $\mathcal{V}$ is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}$ by the $\mathbf{Z}$-action generated by

$$(z,\xi) \mapsto (\lambda^p z,\ e(z)\xi),$$

where $\mathbf{C}^* = \mathbf{C}/2\pi i\mathbf{Z}$ and $e : \mathbf{C}^* \to \mathbf{C}^*$ is the map induced by e_1. If additionally $\mathcal{V}$ has degree 0 then we can take e to be a constant μ. In this case we denote the corresponding degree 0 bundle by $\mathcal{V}_\mu$ and the assignment $\mu \to \mathcal{V}_\mu$ defines an isomorphism of $T' = \mathbf{C}^*/\langle\lambda^p\rangle$ with $\mathrm{Pic}^0(T')$.

Lemma 6.1.3. *The map $\rho : \mathbf{C}^* \times \mathbf{Z}_p \to T'$ of (6.1.2) is given by $\rho(\mu, r) = \mu\lambda^{-r}$.*

Proof of 6.1.3: On restriction to a fibre of π the bundle $\mathcal{V}(\mu, r)$ is of degree 0 with constant multipliers $e_1 = \mu$ and $e_2 = e^{2\pi i r/p}$ corresponding to λ and $e^{2\pi i/p}$ respectively. This bundle is isomorphic to the one with multipliers $e_1 = \mu\lambda^{-r}$ and $e_2 = 1$, giving the result. $\boxed{6.1.3}$

The Hermitian metric defined by (1.1.4) descends to X_p where its associated $(1,1)$ form ω is $\partial\bar\partial$-closed so we can again define the degree of a coherent sheaf over X_p and the notion of stability. Up to a positive scalar factor, the degree of the bundle $\mathcal{V}(\mu, r)$ is given by $\log|\mu|/\log\lambda$. We normalise the degree so that $\deg \mathcal{O}_{X_p}(1) = 1$.

6.2. Holomorphic SL(2, ℂ) bundles

By (2.5.2), a holomorphic $\mathrm{SL}(2,\mathbf{C})$ bundle $\mathcal{E}$ over X_p is of type (3) on at most a finite number of fibres of π, and once again we may define the divisor $D(\mathcal{E})$ of $\mathcal{E}$, which will be a divisor on $\mathbf{P}^1 \times T'/\mathbf{Z}_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$ of bidegree $(k,1)$, where $k = c_2(\mathcal{E})$. Denoting the moduli space of stable $\mathrm{SL}(2,\mathbf{C})$ bundles $\mathcal{E}$ on X_p with $c_2(\mathcal{E}) = k$ by $\mathcal{M}_k(X_p)$ or usually simply by $\mathcal{M}_k$, we again have a holomorphic map:

$$D : \mathcal{M}_k \to \mathbf{P}H^0(\mathcal{O}(k,1)) \cong \mathbf{P}^{2k-1}.$$

One can carry out a similar analysis of the map D as we made in Chapter 3 and Chapter 5. We describe in turn each of the two strata $\mathcal{M}_1^1$ and $\mathcal{M}_1^0$ of $\mathcal{M}_1$ consisting of isomorphism classes of bundles whose graphs are those of degree 1 and degree 0 maps respectively.

6.2.1. Upper stratum

Firstly let us consider a bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 1$ whose divisor is the graph of a degree 1 rational map $\alpha : \mathbf{P}^1 \to \mathbf{P}^1$. The bundle $(\alpha^{-1})^*\mathcal{E}$ has divisor Δ, the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$, so it is enough to be able to construct bundles $\mathcal{E}$ with $D(\mathcal{E}) = \Delta$. The construction of Section 5.5 goes through entirely analogously.

The pull-back $\overline{X}_p$ of X_p to T' via the hyperelliptic quotient $q : T' \to \mathbf{P}^1$ defines a double cover $\overline{\nu} : \overline{X}_p \to X_p$, ramified at the four fibres of : $\overline{X}_p \to T'$ over the half-periods $\pm 1, \pm\sqrt{\lambda^p}$ of T'. The bundle $\overline{\nu}^*\mathcal{E}$ has divisor $\gamma + \iota^*\gamma$, where γ is the graph of the identity map of T' and ι is the involution on $\overline{X}_p$ defined by the double cover. By (2.2.8) and the analogue of (5.4.14), there is a line bundle $\mathcal{L} \in \mathrm{Pic}^0(\overline{X}_p, T')$ with graph γ such that $\overline{\nu}_*\mathcal{L} \cong \mathcal{E}$. Moreover, the analogues of (5.4.3) and (5.4.4) hold; i.e., if $\mathcal{U} \in \mathrm{Pic}^0(T')$, then $\overline{\nu}_*(\mathcal{L} \otimes \overline{\pi}^*\mathcal{U})$ is an SL(2, $\mathbf{C}$) bundle on X_p and is isomorphic to $\mathcal{E}$ if and only if $\mathcal{U} \cong \mathcal{O}_{T'}$. Let $\mathbf{T}$ denote the subset $\{\mathcal{L} \otimes \overline{\pi}^*\mathcal{U} \in \mathrm{Pic}(X_p) : \mathcal{U} \in \mathrm{Pic}^0(T')\}$ of $\mathrm{Pic}(X_p)$.

Once more we may describe the moduli space $\mathcal{M}_1^1$ and the action on it induced by the principal T'-action on X_p in an explicit fashion. The space $\overline{X}_p$ is isomorphic to the quotient of the principal $\mathbf{C}^*$-bundle associated to the line bundle $q^*\mathcal{O}_{\mathbf{P}^1}(-p)$ over T'. This bundle is isomorphic to one given by the quotient of $\mathbf{C}^* \times \mathbf{C}$ described in Section 6.1 for some multiplier $e : \mathbf{C}^* \to \mathbf{C}^*$. The bundle $\overline{X}_p$ is then isomorphic to the principal T'-bundle over T' given by the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the $\mathbf{Z} \times \mathbf{Z}$-action defined by

$$\begin{aligned}(1,0).(z,w) &= (\lambda^p z, e(z)w);\\ (0,1)(z,w) &= (z, \lambda^p w),\end{aligned}$$

and projection defined by $\langle(z,w)\rangle \mapsto \langle z\rangle \in T' = \mathbf{C}^*/\langle\lambda^p\rangle$.

As in Chapter 5, we can now define a bundle $\mathcal{L}_0$ over $\widetilde{X}_p$ as the quotient of $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$ by the action of $\mathbf{Z} \times \mathbf{Z}$ generated by

$$\begin{aligned}(1,0).(z,w,\xi) &= (\lambda^p z, e(z)w, w\xi);\\ (0,1).(z,w,\xi) &= (z, \lambda^p w, z\xi).\end{aligned}$$

The graph of $\mathcal{L}_0$ is clearly γ and pulling back by the (principal) action of $\mu \in T'$ on $\overline{X}_p$ sends $\mathcal{L}_0$ to $\mathcal{L}_0 \otimes \overline{\pi}^*\mathcal{V}_\mu$, where $\mathcal{V}_\mu \in \mathrm{Pic}^0(T')$ is the bundle defined in Section 6.1. Hence the action of $\mu \in T'$ on $\mathbf{T}$ is given by tensoring with the class of $\overline{\pi}^*\mathcal{V}_\mu$. We again have a holomorphic map

$$\begin{aligned}\Theta : \mathrm{Aut}_0(X_p) \times \mathbf{T} &\to \mathcal{M}_1^1;\\ \Theta(\alpha, \mathcal{L}') &= \alpha^*\overline{\nu}_*\mathcal{L}',\end{aligned}$$

for which

$$\Theta(\alpha_1, \mathcal{L}_1) \cong \Theta(\alpha_2, \mathcal{L}_2)$$

if and only if

$$\alpha_1 = R_\mu \circ \alpha_2 \text{ and } \mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathcal{V}_\mu^{-1},$$

giving a holomorphic bijection

$$\overline{\Theta} : \mathrm{Aut}_0(X_p) = \mathrm{Aut}_0(X_p) \times_{T'} \mathbf{T} \longrightarrow \mathcal{M}_1^1,$$

such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Aut}_0(X_p) \arrow[rr, "\overline{\Theta}"] \arrow[dr] & & \mathcal{M}_1^1 \arrow[dl, "D"] \\ & \mathrm{Aut}(\mathbf{P}^1) & \end{tikzcd}$$

The map Θ is equivariant with respect to the action of the subgroup T' of $\mathrm{Aut}_0(X_p)$ leading to the following analogue of (5.5.8):

Theorem 6.2.1. *The natural action of T' on the moduli space $\mathcal{M}_1^1$ is free, giving $D : \mathcal{M}_1^1 \longrightarrow \mathrm{Aut}(\mathbf{P}^1)$ the structure of a principal T'-bundle, diffeomorphic as a principal T'-bundle to $\mathrm{Aut}_0(X_p) \to \mathrm{Aut}(\mathbf{P}^1)$.*

6.2.2. Lower stratum

If $\mathcal{E}$ represents a class in $\mathcal{M}_1^0$ with maximal destabilising line bundles $\mathcal{L}_1, \mathcal{L}_2$ as in Section 3.1, then $\mathcal{E}$ can be written as an extension

6.2.2. $$0 \to \mathcal{L}_i \to \mathcal{E} \to \mathcal{L}_i^{-1}\mathcal{J}_{Z_i} \to 0,$$

where Z_i is the cluster determined by the maximal ideal of a point $x_i \in X_p$. The analogue of (3.4.11) gives $\mathcal{L}_1 \cong \mathcal{L}_2^{-1}\mathcal{O}_{X_p}(-1)$ if $\mathcal{L}_1$ does not restrict to a half-period on the fibres of π. Otherwise $\mathcal{L}_1 \cong \mathcal{L}_2$. Corollaries (3.4.12) and (3.6.3) show that $\deg \mathcal{L}_1 + \deg \mathcal{L}_2 = -1$, so $\mathcal{E}$ is stable if and only if

6.2.3. $$-1 < \deg \mathcal{L}_i < 0 \text{ for } i = 1, 2.$$

The points x_1, x_2 are related by $x_2 = x_1.(\mathcal{L}_1^2)$, where $(\mathcal{L}_1^2)$ denotes the element of $T' \cong \mathrm{Pic}^0(T')$ defined by $\mathcal{L}_1^2$.

Given a line bundle $\mathcal{L}$ on X_p satisfying (6.2.3), the Riemann-Roch formula shows that $H^1(\mathcal{L}^2) = H^2(\mathcal{L}^2) = 0$. The fundamental exact sequence (3.3.5) then shows that a locally free extension $\mathcal{E}$ of the form (6.2.2) exists and is unique up to isomorphism. We can now follow the construction of Section 3.7 to find the diffeomorphism type of $\mathcal{M}_1^0$, but we need to take some care since $\mathrm{Pic}(X_p)$ is not connected.

Recall that the degree of the line bundle $\mathcal{V}(\mu, 1)$ is given by $\log|\mu|/\log\lambda$. Hence the subset $\mathbf{A}$ of $\mathrm{Pic}(X_p) \cong \mathbf{C}^* \times \mathbf{Z}_p$ consisting of line bundles $\mathcal{L}$ satisfying (6.2.3) is given by:

$$\mathbf{A} = \{(\mu, r) : \lambda^{-1} < |\mu| < 1\}.$$

The quotient map

$$\mathrm{Pic}(X_p) \to T' = \mathrm{Pic}(X_p)/\tilde{\pi}^*(\mathrm{Pic}(\mathbf{P}^1)),$$

can be re-expressed using the isomorphisms $\mathrm{Pic}(X_p) \cong \mathbf{C}^* \times \mathbf{Z}_p$ and $T' \cong \mathbf{C}^*/\{\lambda^p\}$ as

$$(\mu, r) \mapsto \mu\lambda^{-r}.$$

Under this map, the set $\mathbf{A}$ maps biholomorphically onto $T' \setminus S$, where S is the set defined by

$$S = \{\langle\mu\rangle \in T' : |\mu| = \lambda^n \text{ for some } n \in \mathbf{Z}\}.$$

The set S consists of p disjoint circles C_i ($i = 1, \ldots, p$) in T', where

$$C_i = \{\langle\mu\rangle \in T' : |\mu| = \lambda^i\}.$$

The circle C_0 passes through the two half-periods ± 1 of T', and when $p = 2m$ is even the circle C_m passes through the other two half-periods $\pm\sqrt{\lambda^p}$. The hyperelliptic involution ι on T' given by $z \mapsto z^{-1}$ maps C_0 to itself via the map $e^{i\theta} \mapsto e^{-i\theta}$, and maps C_i to C_{k-i} via the map $\lambda^k e^{i\theta} \mapsto \lambda^{p-k}e^{-i\theta}$ for $i = 1, \ldots, [p/2]$.

Geometrically, ι corresponds to rotation of π about the axis through the half-periods of T'. In the quotient $T'/\mathbf{Z}_2 \cong \mathbf{P}^1$, the image of C_0 is homeomorphic to a closed interval I_0 with endpoints $q(\pm 1)$. If $p = 2m$ is even, the image of C_m is likewise a closed interval I_1 with endpoints $q(\pm\sqrt{\lambda^p})$. Otherwise the circles C_i and C_{p-i} are mapped diffeomorphically onto a circle S_i for $i = 1, \ldots, m-1$. When $p = 2m+1$ is odd, the circles C_i and C_{p-i} are mapped pairwise onto circles S_i for $i = 1, \ldots, m$ in exactly the same fashion. Let Σ_p denote the union of disjoint subsets of $\mathbf{P}^1$ given by:

$$\begin{aligned} I_0 \sqcup I_1 \sqcup S_1 \sqcup \ldots \sqcup S_{m-1}, &\quad (p = 2m);\\ I_0 \sqcup S_1 \sqcup \ldots \sqcup S_m, &\quad (p = 2m+1). \end{aligned}$$

An element $\mathcal{E}$ of $\mathcal{M}_1^0$ has a divisor of the form

6.2.4. $$[X_0, X_1] \times \mathbf{P}^1 + \mathbf{P}^1 \times [L_0, L_1],$$

for some point $([X_0, X_1], [L_0, L_1]) \in \mathbf{P}^1 \times \mathbf{P}^1$. We may think of $\mathbf{P}^3$ as $\mathbf{P}M(2,\mathbf{C})$, where $M(2,\mathbf{C})$ denotes the set of 2×2 complex matrices. using the identification

6.2.5. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [a, b, c, d].$$

Under this identification, the divisor of a bundle in $\mathcal{M}_1^1$ is a point of $\mathrm{PGL}(2,\mathbf{C})$, with the corresponding automorphism of $\mathbf{P}^1$ given by the map

$$[X_0, X_1] \mapsto [aX_0 + bX_1, cX_0 + dX_1].$$

A divisor of the form (6.2.4) corresponds to the element

6.2.6. $$\begin{pmatrix} L_0X_1 & -L_0X_0 \\ L_1X_1 & -L_1X_0 \end{pmatrix} \in \mathrm{PGL}(2,\mathbf{C}),$$

defining an embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}M(2,\mathbf{C})$ whose image is the quadric hypersurface defined by the vanishing of the determinant. The above discussion shows that divisors of classes in $\mathcal{M}_1^0$ are represented by points $(x, \mathcal{L}) \in \mathbf{P}^1 \times (\mathbf{P}^1 \setminus \Sigma_p) \subset \mathbf{P}M(2,\mathbf{C})$.

We then have the following analogue of (3.7.5) for $S^1 \times L_p$.

Theorem 6.2.7. *The image of D restricted to $\mathcal{M}_1^0$ is $\mathbf{P}^1 \times (\mathbf{P}^1 \setminus \Sigma_p)$. The natural action of T' on $\mathcal{M}_1^0$ is free, making*

$$D : \mathcal{M}_1^0 \to \mathbf{P}^1 \times (\mathbf{P}^1 \setminus \Sigma_p)$$

into a principal T'-bundle.

Proof of 6.2.7: The image of the set $\mathbf{A} \subset \mathbf{C}^* \times \mathbf{Z}_p$ under the successive quotients

$$\mathbf{C}^* \times \mathbf{Z}_p \xrightarrow{\rho} T' \xrightarrow{q} \mathbf{P}^1$$

is the subset $\mathbf{P}^1 \setminus \Sigma_p$. The map ρ is injective on restriction to $\mathbf{A}$ and under the composition $\rho' = q \circ \rho|_{\mathbf{A}}$, two distinct points (μ, r) and (ν, s) of $\mathbf{A}$ have the same image in $\mathbf{P}^1$ if and only if:

6.2.8. $$\mu\nu = \lambda \text{ and } r + s = 1.$$

The Serre construction gives a surjective holomorphic map

$$\Theta : \mathbf{A} \times X_p \to \mathcal{M}_1^0,$$

invariant under the $\mathbf{Z}_2$-action generated by

$$((\mu, r), x) \mapsto ((\mu^{-1}\lambda, 1 - r), x.(\mu\lambda^{-r})^2).$$

The map Θ fits into a commutative diagram:

$$\begin{tikzcd}
\mathbf{A} \times X_p \arrow[rr, "\Theta"] \arrow[dr, "\rho' \times \pi"'] & & \mathcal{M}_1^0 \\
& (\mathbf{P}^1 \setminus \Sigma_p) \times \mathbf{P}^1 & \arrow[ul, "D"]
\end{tikzcd}$$

which is equivariant under the T'-action. When p is even the $\mathbf{Z}_2$-action on $\mathbf{A}$ is free. When p is odd, the $\mathbf{Z}_2$-action is trivial on the fibres $\{a\} \times X_p$ of $\mathbf{A} \times X_p$ over the fixed points $a = (\pm\sqrt{\lambda^p}, (1-p)/2)$ of the $\mathbf{Z}_2$-action on $\mathbf{A}$. In either case we obtain a well-defined fibration with fibre X_p:

$$\mathbf{B} = (\mathbf{A} \times X_p)/\mathbf{Z}_2 \longrightarrow \mathbf{A}/\mathbf{Z}_2 \cong \mathbf{P}^1 \setminus \Sigma_p.$$

The T'-action on $\mathbf{A} \times X_p$ commutes with the $\mathbf{Z}_2$-action, descending to a principal action on $\mathbf{B}$. We then have a T'-equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{B} \arrow[rr, "\overline{\Theta}"] \arrow[dr] & & \mathcal{M}_1^0 \\
& (\mathbf{P}^1 \setminus \Sigma_p) \times \mathbf{P}^1 & \arrow[ul, "D"]
\end{tikzcd}$$

where $\overline{\Theta}$ is a holomorphic bijection, from which the result follows. $\boxed{6.2.7}$

Combining (6.2.1) and (6.2.7) gives:

Theorem 6.2.9. *The map $D : \mathcal{M}_1 \to \mathbf{P}^3 \setminus (\mathbf{P}^1 \times \Sigma_p)$ is a principal T'-bundle.*

6.3. The charge 1 moduli space

We now investigate the extent to which we can determine the diffeomorphism type of $\mathcal{M}_1$ given it is a principal T'-bundle. Any principal T'-bundle Y over a base manifold B has only one topological invariant given as follows. The total space of Y is isomorphic to the quotient of the principal $\mathbf{C}^*$-bundle associated to a complex line bundle $\mathcal{L}$ over B and Y is then classified up to diffeomorphism by $c_1(\mathcal{L}) \in H^2(B;\mathbf{Z})$. Hence, in our case we are interested in the integral cohomology of the spaces $\mathbf{P}^3 \setminus (\mathbf{P}^1 \times \Sigma_p)$ in degree 2. Recall that if we remove from $\mathbf{P}^3$ the line l_∞ given in homogeneous coordinates by

$$l_\infty = \{[z_0, z_1, z_2, z_3] : (z_0, z_1) = 0\},$$

the resulting space is isomorphic to the total space of the rank 2 holomorphic vector bundle $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ over $\mathbf{P}^1$ with projection map φ given by

$$\varphi([z_0, z_1, z_2, z_3]) = [z_0, z_1].$$

Although $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ is non-trivial, even when regarded as a complex bundle, it is trivial when regarded a smooth real bundle. This is because rank 2 complex vector bundles over $\mathbf{P}^1$ are classified by $\pi_1(\mathrm{U}(2)) \cong \mathbf{Z}$, whereas (orientable) rank 4 real bundles are classified by $\pi_1(\mathrm{SO}(4)) \cong \mathbf{Z}_2$. Under the natural map $\pi_1(\mathrm{U}(2)) \to \pi_1(\mathrm{SO}(4))$ induced by the inclusion $\mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$ the class of $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ is mapped to 0 [4, p.41].

Fix a class $\mathcal{L}_0 \in I_0 \subset \mathbf{P}^1$ and choose coordinates on $\mathbf{P}^1$ so that $\mathcal{L}_0$ is given by $[0,1]$. Under the identifications (6.2.6) and (6.2.5), a divisor

6.3.1. $$[X_0, X_1] \times \mathbf{P}^1 + \mathbf{P}^1 \times [L_0, L_1]$$

corresponds to the point

6.3.2. $$[L_0X_1, -L_0X_0, L_1X_1, -L_1X_0] \in \mathbf{P}^3.$$

In particular, the set of divisors $\{\{x\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\mathcal{L}_0\} . x \in \mathbf{P}^1\}$ corresponds to the line l_∞ in $\mathbf{P}^3$. Now consider a divisor of the form (6.3.1) with $L_0 \neq 0$. Using (6.3.2), the map $\varphi : \mathbf{P}^3 \setminus l_\infty \to \mathbf{P}^1$ sends this point to $[X_1, -X_0] \in \mathbf{P}^1$. Hence, as $[L_0, L_1]$ varies through $\mathbf{P}^1 \setminus \{[0,1]\}$, (6.3.2) defines a map $s : \mathbf{P}^1 \setminus \{[0,1]\} \to H^0(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ given by

$$s_L([X_1, -X_0]) = [X_1, -X_0, LX_1, -LX_0],$$

where $L = L_1/L_0$. Any two sections s_L and $s_{L'}$ differ by a scalar multiple and, since if $L \neq 0$ the section s_L is nowhere vanishing, the image of s generates a trivial holomorphic line sub-bundle $\mathcal{U}$ of $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. The map s then gives an isomorphism of the space of divisors

6.3.3. $$\mathbf{P}^1 \times (\Sigma_p \setminus \{\mathcal{L}_0\}) = \{\{x\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\mathcal{L}\} : x \in \mathbf{P}^1, \mathcal{L} \in \Sigma_p \setminus \{\mathcal{L}_0\}\}$$

with a subset $\tilde{\Sigma}_p$ of $\mathcal{U}$. Since the value of any holomorphic section of $\mathcal{U}$ is uniquely determined by its evaluation at a fixed point $x_0 \in \mathbf{P}^1$, evaluating at x_0 gives a diffeomorphism of the space $\mathcal{U} \setminus \tilde{\Sigma}_p$ with $\mathbf{P}^1 \times (\mathbf{C} \setminus \Sigma_p')$, where Σ_p' is a disjoint union of sets:

$$\tilde{I}_0' \sqcup I_1' \sqcup S_1' \sqcup \ldots \sqcup S_{m-1}', \; (p = 2m)$$
$$\tilde{I}_0' \sqcup S_1' \sqcup \ldots \sqcup S_m', \; (p = 2m + 1)$$

The set $\tilde{I}_0'$ is the image of $I_0 \setminus \{\mathcal{L}_0\}$ under evaluation at x_0 and is homeomorphic to the half-open interval $[0, 1)$. The sets I_1', S_j' are the images of I_1, S_j under evaluation respectively.

Since $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ is trivial as a real bundle, we may choose a trivial real complement $\mathcal{U}'$ for $\mathcal{U}$. Hence $\mathbf{P}^3 \setminus (\mathbf{P}^1 \times \Sigma_p)$ is diffeomorphic to the space

$$\mathcal{U}' \oplus (\mathcal{U} \setminus \tilde{\Sigma}_p) \cong \mathbf{P}^1 \times (\mathbf{R}^4 \setminus \Sigma_p').$$

We can determine the cohomology of $\mathbf{R}^4 \setminus \Sigma_p'$ using excision and a Mayer-Vietoris argument as follows. First note that $\mathbf{R}^4 \setminus \tilde{I}_0'$ is homotopy equivalent to $\mathbf{R}^4$. Hence $\mathbf{R}^4 \setminus \Sigma_p'$ is obtained from $\mathbf{R}^4$ (up to homotopy) by removing the following collections of pairwise disjoint subsets:

(a) a closed interval and $(m-1)$ circles in the case where $p = 2m$, or

(b) m circles in the case $p = 2m + 1$.

Let A denote an open set in $\mathbf{R}^4$ consisting of pairwise disjoint neighbourhoods of the subsets listed above, diffeomorphic to D^4 in the case of the interval and $S^1 \times D^3$ in the case of the circles, where D^n denotes the open n-disc. Let Y denote $\mathbf{R}^4 \setminus \Sigma_p'$, so $\mathbf{R}^4 = Y \cup A$ and in each case $Y \cap A$ consists up to homotopy of the disjoint unions:

(a) $S^3 \sqcup \underbrace{(S^1 \times S^2) \sqcup \ldots \sqcup (S^1 \times S^2)}_{(m-1)\text{ copies}}$;

(b) $\underbrace{(S^1 \times S^2) \sqcup \ldots \sqcup (S^1 \times S^2)}_{m\text{ copies}}$.

Applying the Mayer-Vietoris sequence for the cohomology of the exact triad $(\mathbf{R}^4, Y, A)$ [29, p.140] we obtain an exact sequence:

$$\to H^q(\mathbf{R}^4; \mathbf{Z}) \to H^q(Y; \mathbf{Z}) \oplus H^q(A; \mathbf{Z}) \to H^q(Y \cap A; \mathbf{Z}) \to$$

which for $q = 2$ gives

(a) $H^2(Y; \mathbf{Z}) \cong \mathbf{Z}^{m-1}$;

(b) $H^2(Y; \mathbf{Z}) \cong \mathbf{Z}^m$

respectively. In the case $q = 1$, we have an exact sequence

$$0 \to H^1(Y; \mathbf{Z}) \oplus H^1(A; \mathbf{Z}) \to H^1(Y \cap A; \mathbf{Z}) \to 0.$$

Since both $H^1(A;\mathbf{Z})$ and $H^1(Y\cap A;\mathbf{Z})$ are free $\mathbf{Z}$-modules with generators given by loops in each copy of $S^1\times S^2$ of the form $S^1\times\{y\}$, with $y\in S^2=\partial D^3$ constant, this gives $H^1(Y;\mathbf{Z})=0$. Hence, by the Künneth theorem we obtain

$$H^2(\mathbf{P}^1\times Y;\mathbf{Z})\cong \pi_1^*H^2(\mathbf{P}^1;\mathbf{Z})\oplus\pi_2^*H^2(Y;\mathbf{Z}),$$

where π_1 and π_2 denote the projections onto each factor. Associated to the moduli space $\mathcal{M}_1$ we therefore have classes $\alpha\in H^2(\mathbf{P}^1;\mathbf{Z})$ and $\beta\in H^2(Y;\mathbf{Z})$. The class α is determined by the restriction of $\mathcal{M}_1$ to any section S of $\mathcal{U}$.

Lemma 6.3.4. *Let S be any section of the fibre bundle $\mathcal{U}\setminus\Sigma_p$, then the space $\mathcal{M}_1|_S$ is diffeomorphic to X_p.*

Proof of 6.3.4: The diffeomorphism type of $\mathcal{M}_1|_S$ is independent of the choice of S so we may choose the section S of the statement to consist of divisors of the form

$$[X_0,X_1]\times\mathbf{P}^1+\mathbf{P}^1\times\{\mathcal{L}\},\ [X_0,X_1]\in\mathbf{P}^1,$$

where $\{\mathcal{L}\}$ is the class in $\mathbf{P}^1\setminus\Sigma_p$ corresponding to a line bundle $\mathcal{L}$ on X_p in the annulus $\mathbf{A}$ of (6.2.7) which does not restrict to a half-period on the fibres of π. The Serre construction defines a holomorphic bijection

$$X_p\to\mathcal{M}_1|_S,$$

given by associating to each point $x\in X_p$ the unique isomorphism class of locally free extension $\mathcal{E}$ defined by

$$0\to\mathcal{L}\to\mathcal{E}\to\mathcal{L}^{-1}\mathcal{J}_x\to 0.$$

$\boxed{6.3.4}$

Corollary 6.3.5. *The class $\alpha\in H^2(\mathbf{P}^1;\mathbf{Z})$ defined above for $\mathcal{M}_1(X_p)$ is given by $-pH$, where H is the standard generator of $H^2(\mathbf{P}^1;\mathbf{Z})$ defined by the natural orientation.*

In order order to completely determine the diffeomorphism type of $\mathcal{M}_1$, we need to determine the class $\beta\in H^2(Y;\mathbf{Z})$. However, in the case where $H^2(Y;\mathbf{Z})=0$, namely when $p=1$ or 2, the space $\mathcal{M}_1(X_p)$ is isomorphic to the pull-back of $\mathcal{M}_1|_{\mathbf{P}^1}$ via the map $\varphi:\mathbf{P}^3\setminus(\mathbf{P}^1\times\Sigma_p)\to\mathbf{P}^1$ and we obtain

Theorem 6.3.6. *For $p=1,2$ there are diffeomorphisms*

$$\mathcal{M}_1(X_p)\cong X_p\times(\mathbf{R}^4\setminus\Sigma_p').$$

Remark 6.3.7. This result for the case $p=1$ was proved in [16].

6.4. Concluding remarks

One might expect Theorem 6.3.6 to hold for all values of p. In order to prove the result in the general case, however, we must show that the obstruction $\beta \in H^2(Y;\mathbf{Z})$ to extending the principal T'-bundle over the set Σ_p actually vanishes. To do so directly, we would have to show that the restriction of $\mathcal{M}_1(X_p)$ to fibres of $\varphi : \mathbf{P}^3 \setminus (\mathbf{P}^1 \times \Sigma_p) \to \mathbf{P}^1$ admits a section. This requires the construction of a holomorphic family of stable $\mathrm{SL}(2,\mathbf{C})$ bundles for which the divisor type must 'jump'. Unfortunately, there seems to be no satisfactory method of producing such a family. Indeed, this was the main reason for introducing the stratification by divisor types.

Alternatively, we could try to enlarge the space of bundles under consideration and prove the resulting space has the desired property. Indeed, there are two natural ways in which we could try to compactify the moduli spaces. The first of these is the Uhlenbeck compactification. The weak compactness theorem of Uhlenbeck [44] and results of Donaldson [21, Chapters 4, 7 and 8] show we can compactify $\mathcal{M}_1$ by adjoining $X_p \times \mathcal{M}_0$, where $\mathcal{M}_0$ is the moduli space of flat $\mathrm{SU}(2)$ connections on X_p. Formally, we adjoin limits of sequences of points "going to infinity" in $\mathcal{M}_1$. Uhlenbeck's theorem shows that given such a sequence, the corresponding curvatures concentrate about a point x of X_p and are very flat away from this point. In the limit a standard 1-instanton on S^4 "bubbles off" leaving the flat connection A over X_p. A point (x, A) of $X_p \times \mathcal{M}_0$ then represents an "idealised connection" whose curvature is a δ-function at $x \in X_p$ and a flat connection A on a trivial $\mathrm{SU}(2)$-bundle over X_p.

It is a well-known result that $\mathcal{M}_0$ is the space $\mathrm{Hom}(\pi_1(X_p), \mathrm{SU}(2))/\mathrm{SU}(2)$ of equivalence classes of representations of $\pi_1(X_p) \cong \mathbf{Z} \times \mathbf{Z}_p$ in $\mathrm{SU}(2)$. This space is easy to calculate (see [1] for background material on representation theory). Since $\pi_1(X_p)$ is Abelian, any representation ρ in $\mathrm{SU}(2)$ takes its values in a maximal torus. Fix a maximal torus $T_0 = \{\mathrm{diag}(\mu, \mu^{-1}) : \mu \in S^1\}$, then after a conjugation, if necessary, we may assume ρ takes its values in T_0. It is then defined uniquely by the point of $T_0 \times \mathbf{Z}_p$ given by its values on the generators of $\mathbf{Z} \times \mathbf{Z}_p$. Denote by ρ_g^r the representation ρ such that $\rho(1,0) = g$ and $\rho(0,1) = e^{2\pi r i/p}$ for some $g \in T_0$, and let $C_j = \{\rho_g^j : g \in T_0\}$. The residual part of the adjoint action of $\mathrm{SU}(2)$ on T_0 is given simply by complex conjugation. Hence, $\rho_g^r \sim \rho_h^s$ if and only if:

$$\begin{aligned}(g, r) &= (h, s)\\ \text{or } (g, r) &= (\overline{h}, p - s)\end{aligned}$$

The fixed points of this action are given by $(\pm 1, 0) \in C_0$ and $(\pm 1, m) \in C_m$, if $p = 2m$ is even. Quotienting out by this residual action, gives

$$\begin{aligned}\mathcal{M}_0 &\cong \sqcup\{I_0, C_1, \ldots, C_m\}, \text{ when } p = 2m + 1.\\ \mathcal{M}_0 &\cong \sqcup\{I_0, I_1, C_1, \ldots, C_{m-1}\}, \text{ when } p = 2m.\end{aligned}$$

The space I_0 is a closed interval and is the image of the circle of representations C_0. When $p = 2m$ is even, the quotient of the circle C_m is the space I_1, and is again a closed interval. Otherwise the remaining circles C_j are identified pairwise. Hence the ends of our moduli spaces correspond with those expected from representation theory. If we could adjoin the set $X_p \times \mathcal{M}_0$ in such a way that the resulting compactification $\mathcal{M}_1^{U}$ was a principal T'-bundle over $\mathbf{P}^3$, the general result would be proved. Unfortunately, such methods are purely differential geometric and it is unclear whether $\mathcal{M}_1^{U}$ can be given the structure of a smooth manifold or how it fits in with the complex analytic methods used above.

A more natural complex analytic way of extending the moduli space would be to consider the moduli space $\mathcal{M}_1^{s.s.}$ of semi-stable bundles. The set $\mathcal{M}_1^{s.s.} \setminus \mathcal{M}_1$ consists of isomorphism classes of bundles $\mathcal{E}$ which are extensions of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}\mathcal{J}_x \to 0,$$

where $\mathcal{L}$ is a line bundle on X_p with $\deg \mathcal{L} = 0$ and $x \in X_p$ is a point. Such line bundles correspond precisely to the set Σ_p defined above and the space of such objects has a natural T'-action. However, in our case deformation theory shows that the moduli space of *all* holomorphic structures on E is a smooth manifold in which the subset $\mathcal{M}_1^{s.s.}$ is not open and so does not carry the natural submanifold structure. It is therefore an unlikely candidate for our purposes.

Finally we remark that the "next step" is to consider elliptic fibrations over $\mathbf{P}^1$ with *multiple fibres*. Examples of such spaces are the products $S^1 \times L(p,q)$ and $S^1 \times \Sigma(p,q,r)$. Here $L(p,q)$ denotes the Lens space obtained by taking the quotient of S^3 by the $\mathbf{Z}_p$-action defined by

$$(z_0, z_1) \mapsto (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1),$$

for a pair of coprime natural numbers p, q. The manifold $\Sigma(p,q,r)$ is the Brieskorn homology 3-sphere defined for triples (p,q,r) of pairwise coprime natural numbers with $r > q > p \geq 2$ (see [37] for details). With their natural structures as elliptic surfaces, the space $S^1 \times L(p,q)$ has one multiple fibre of type ${}_qI_0$, whilst $S^1 \times \Sigma(p,q,r)$ has three multiple fibres of types ${}_pI_0$, ${}_qI_0$ and ${}_rI_0$. One expects much of the details of the previous construction to carry through for these spaces. Indeed the Jacobian variety is again trivial. The most significant difference which the appearance of multiple fibres seems to make is in the canonical bundle, which is no longer just the pull-back of the corresponding object on $\mathbf{P}^1$, but has correction terms depending on the multiplicities of the fibres [12]. This in turn will affect the relationship between the maximal destabilising line bundles and thus we would expect this additional structure to be primarily reflected in the lower stratum of the moduli space, and in particular, of course, in its ends.

Chapter 7

Twistorial methods

7.1. Twistor spaces and the Ward correspondence

We shall review very briefly some background material on twistor spaces and their rôle in the study of the anti-self-duality equations, principally to set up the notation and provide the motivation for the rest of this chapter. For more detailed information on the theory of twistor spaces we refer to [8],[41] and [47].

If M is an oriented Riemannian 4-manifold whose metric is anti-self-dual (i.e., the self-dual part of its Weyl tensor vanishes), then the projective $+1/2$ spinor bundle $\mathbf{P}(V_+)$ of M has a natural integrable complex structure. The resulting complex 3-manifold Z is called the *twistor space* of M. The bundle projection defines a smooth map

$$P : Z \to M,$$

whose fibres are isomorphic to the projective line $\mathbf{P}^1$ and are holomorphically embedded in Z, with normal bundles all isomorphic to $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. For each $x \in M$, the fibre $P^{-1}(x)$ of P over x parameterises in a natural way all the almost complex structures on the tangent space of M at x which are compatible with both the metric and orientation. The quaternionic structure on $\mathbf{P}(V_+)$ defines an anti-holomorphic involution $\tau : Z \to Z$ called the *real structure*. Although τ has no fixed points, it does preserve the fibres of P, which are consequently often called the *real lines*. We shall usually refer to them as *twistor lines*.

When M is spin the tautological line bundle ξ over $\mathbf{P}(V_+)$ is well-defined and has a natural holomorphic structure. Furthermore, the bundle ξ is isomorphic to $\overline{\tau^*\xi}$, i.e., ξ is *real*, and the canonical bundle $\mathcal{K}_Z$ of Z is isomorphic to ξ^{-4}. For each $x \in M$ the bundle ξ restricts to $\mathcal{O}_{\mathbf{P}^1}(-1)$ on $P^{-1}(x) \cong \mathbf{P}^1$. Hence we shall denote ξ by $\mathcal{O}_Z(-1)$ when it exists.

We now recall the *Ward correspondence* which relates anti-self-dual connections on an SU(2) bundle F over M to certain holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles over Z. (See [4], [8], [47] for more details). If A is an anti-self-dual connection on F then its pull-back on the bundle $E = P^*F$ has curvature of type

$(1,1)$ and so defines a holomorphic structure $\mathcal{E}$ on E (c.f. Section 1.3). The holomorphic bundle $\mathcal{E}$ has the following properties:

(i) The bundle $\mathcal{E}|P^{-1}(x)$ is holomorphically trivial for all points $x \in M$.

(ii) $\det \mathcal{E} \cong \mathcal{O}_Z$.

(iii) For each $z \in Z$ there is an isomorphism $\theta : \mathcal{E} \to \overline{\tau^* \mathcal{E}^*}$ such that

$$(e_1, \theta(e_2)) = \overline{(e_2, \theta(e_1))}, \text{ for all } e_1 \in \mathcal{E}_z, e_2 \in \mathcal{E}_{\tau(z)},$$

where $(\cdot,\cdot)$ denotes the natural pairing of $\mathcal{E}$ and $\mathcal{E}^*$, such that θ induces a positive definite Hermitian metric on the vector spaces $H^0(P^{-1}(x); \mathcal{E})$ for all $x \in M$.

We are now ready to state the following theorem of Atiyah, Hitchin and Singer [8], which develops the ideas of Ward [46].

Theorem 7.1.1 (The Ward Correspondence). *Let M be an anti-self-dual Riemannian 4-manifold and let Z be its twistor space. There is a natural one-to-one correspondence between isomorphism classes of holomorphic rank 2 bundles $\mathcal{E}$ on Z satisfying properties (i) – (iii) above and gauge equivalence classes of anti-self-dual connections on* SU(2) *bundles over M.*

Remark 7.1.2. There are versions of the Ward correspondence for groups other than SU(2). See [9].

We make the following definition.

Definition 7.1.3. A holomorphic rank 2 bundle $\mathcal{E}$ over Z which satisfies conditions (i) – (iii) above is called an *instanton bundle*. If the corresponding bundle F on M has $c_2(F) = k$, we say that $\mathcal{E}$ is a k-instanton bundle.

Thus, in principle, if we can construct instanton bundles on Z then we can recover solutions to the anti-self-duality equations over M. The case when $M = S^4$ was investigated by Atiyah and Ward in [9]. Identifying S^4 with the quaternionic projective line $\mathbf{HP}^1$, its twistor space is $\mathbf{CP}^3$ with the twistor fibration P given in homogeneous coordinates by

$$P([z_0, z_1, z_2, z_3]) = [z_0 + z_1 j, z_2 + z_3 j].$$

Let $\mathcal{O}(-1)$ denote the tautological line bundle on $\mathbf{CP}^3$. Its dual $\mathcal{O}(1)$ is ample, hence given any holomorphic rank 2 bundle $\mathcal{E}$ on $\mathbf{CP}^3$, for some $n \in \mathbf{Z}$ the bundle $\mathcal{E} \otimes \mathcal{O}(n)$ admits a non-trivial section vanishing on a codimension 2 subscheme Y of Z, as in Section 2.4. The bundle $\mathcal{E}$ then fits into an exact sequence

$$0 \to \mathcal{O}(-n) \to \mathcal{E} \to \mathcal{O}(n)\mathcal{J}_Y \to 0.$$

Thus solutions to the anti-self-duality equations over S^4 can *all* be obtained via the Serre construction, providing we choose the curve Y and the extension class suitably such that the descent conditions (i) – (iii) are satisfied. In the case when Y is a sum of twistor lines the solutions obtained are the well-known 't Hooft solutions [8, 9]. Atiyah and Ward show that in the cases where $c_2(F) = 1$ or 2 *all* solutions can be obtained by the 't Hooft construction. For $c_2(F) \geq 3$, however, one generically has to use curves other then twistor lines.

The metric on $S^1 \times S^3$ defined in Section 1.1 is conformally flat and so anti-self-dual. The twistor space therefore exists but, as we shall see, it contains few curves. Consequently, even in the charge 1 case, the instanton solutions on $S^1 \times S^3$ which can be obtained by the 't Hooft/Atiyah-Ward construction are non-generic. In the charge 1 case the generic solution can, however, be constructed by a method based partly on a 3-dimensional version of the Friedman construction.

7.2. The twistor space of $S^1 \times S^3$.

The twistor space of $\mathbf{R}^4 \setminus \{0\}$ is given by $\mathbf{P}^3 \setminus (l_0 \cup l_\infty)$, where l_0, l_∞ are the projective lines defined by

$$l_0 = \{[z_0, z_1, z_2, z_3] \in \mathbf{P}^3 : (z_0, z_1) = 0\};$$
$$l_\infty = \{[z_0, z_1, z_2, z_3] \in \mathbf{P}^3 : (z_2, z_3) = 0\}.$$

Identifying $\mathbf{R}^4 \setminus \{0\}$ with $\mathbf{H}^*$ in the usual manner, the twistor fibration P is given by

$$\begin{aligned} P : \mathbf{P}^3 \setminus (l_0 \cup l_\infty) &\longrightarrow \mathbf{R}^4 \setminus \{0\} \\ [z_0, z_1, z_2, z_3] &\longrightarrow (z_2 + z_3 j)^{-1}(z_0 + z_1 j), \end{aligned} \qquad \textbf{7.2.1.}$$

and the real structure τ by

7.2.2. $$\tau([z_0, z_1, z_2, z_3]) = [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2].$$

Let $Q = \mathbf{P}_0^1 \times \mathbf{P}_\infty^1$, where $\mathbf{P}_0^1$ and $\mathbf{P}_\infty^1$ are both copies of $\mathbf{P}^1$, and define a map $\pi : \mathbf{P}^3 \setminus (l_0 \cup l_\infty) \to Q$ by

$$\pi : ([z_0, z_1, z_2, z_3]) = ([z_0, z_1], [z_2, z_3]).$$

The group $\mathbf{C}^*$ acts fibrewise with respect to π, the action of $\mu \in \mathbf{C}^*$ being given by

7.2.3. $$\mu.[z_0, z_1, z_2, z_3] = [\mu^{1/2} z_0, \mu^{1/2} z_1, \mu^{-1/2} z_2, \mu^{-1/2} z_3].$$

This action is free and transitive on the fibres so $\pi : \mathbf{P}^3 \setminus (l_0 \cup l_\infty) \to Q$ is a principal $\mathbf{C}^*$-bundle. It is easy to check using local trivialisations that it is the principal $\mathbf{C}^*$-bundle associated to the line bundle $\mathcal{O}_Q(1, -1)$ over Q.

Recall that our standard model for $S^1 \times S^3$ is the manifold obtained by taking the quotient of $\mathbf{H}^*$ by the $\mathbf{Z}$-action generated by the conformal transformation $z \mapsto \lambda z$, where $\lambda \in \mathbf{R}$ is fixed with $\lambda > 1$. This action lifts to a free holomorphic action on $\mathbf{P}^3 \setminus (l_0 \cup l_\infty)$ with generator

$$[z_0, z_1, z_2, z_3] \mapsto [\lambda^{1/2} z_0, \lambda^{1/2} z_1, \lambda^{-1/2} z_2, \lambda^{-1/2} z_3].$$

The quotient $Z = (\mathbf{P}^3 \setminus (l_0 \cup l_\infty))/\mathbf{Z}$ is the twistor space of $S^1 \times S^3$. Since $\mathbf{Z}$ acts as a subgroup of $\mathbf{C}^*$, the map π defines a principal elliptic fibration (which we also denote by π),

$$\pi : Z \to Q,$$

with fibres all isomorphic to the elliptic curve $T = \mathbf{C}^*/\langle\lambda\rangle$.

As λ is real, the twistor fibration $P : Z \to S^1 \times S^3$ and real structure $\tau : Z \to Z$ are the obvious ones induced by (7.2.1) and (7.2.2), and there is an obvious real structure τ' on Q such that $\tau' \circ \pi = \pi \circ \tau$. The restriction of π to a twistor line gives an isomorphism with a smooth τ'-invariant $(1,1)$ curve on Q.

If we denote the projections of Z to $\mathbf{P}^1_0$ and $\mathbf{P}^1_\infty$ by π_0 and π_∞ respectively, then the fibres $X_p = \pi_0^{-1}(p)$, $X'_q = \pi_\infty^{-1}(q)$ for $p \in \mathbf{P}^1_0$, $q \in \mathbf{P}^1_\infty$, are all isomorphic to our standard Hopf surface X. The map $P|X_p$ is a diffeomorphism of X_p with $S^1 \times S^3$, and if $p = [p_0, p_1]$, the complex structure on $S^1 \times S^3$ induced by that on Z via this diffeomorphism is the one given by the quaternion

$$(p_0 + p_1 j)^{-1} i (p_0 + p_1 j) \in \mathbf{S}.$$

The situation for X'_q is similar.

Remark 7.2.4. The transverse families of Hopf surfaces will be important in the sequel. To avoid the notation becoming cumbersome we shall sometimes refer simply to X_p or X'_q when we mean a generic element of the respective family.

Since $S^1 \times S^3$ is spin, the $+1/2$ spinor bundle V_+ exists globally and is trivial as a real bundle. Hence Z is diffeomorphic to $S^1 \times S^3 \times S^2$, giving $H^2(Z;\mathbf{Z}) \cong \mathbf{Z}$. The tautological line bundle $\mathcal{O}_Z(-1)$ over $Z \cong \mathbf{P}(V_+)$ exists and $K_Z \cong \mathcal{O}_Z(-4)$. The holomorphic invariants of Z are summarised in the following lemma.

Lemma 7.2.5. *The following equalities hold:*

$$h^1(Z;\mathcal{O}_Z) = 1; \quad h^2(Z;\mathcal{O}_Z) = 0; \quad h^3(Z;\mathcal{O}_Z) = 0.$$

Proof of 7.2.5: Apply the Leray spectral sequence with

$$E_2^{p,q} = H^p(Q; R^q\pi_*\mathcal{O}_Z).$$

Since T acts trivially on $H^1(\pi^{-1}(x);\mathcal{O}_Z)$ for all $x \in Q$ we have $R^1\pi_*\mathcal{O}_Z \cong \mathcal{O}_Q$. This and the fact that $\pi_*\mathcal{O}_Z \cong \mathcal{O}_Q$ implies the vanishing of all terms $E_2^{p,q}$ with $p + q = 2$, and so $h^2(Z;\mathcal{O}_Z) = 0$. The vanishing of $h^3(Z;\mathcal{O}_Z)$ follows by Serre duality. An application of Riemann-Roch then gives $H^1(Z;\mathcal{O}_Z) = 1$. $\boxed{7.2.5}$

Taking the long exact sequence of cohomology induced by the exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_Z \to \mathcal{O}^*_Z \to 0,$$

shows that $\mathrm{Pic}(Z) \cong \mathbf{C}^* \times \mathbf{Z}$. The bundle $\mathcal{O}_Z(1)$ defines a generator for the $\mathbf{Z}$ factor, and we can define a bundle $\mathcal{V}_\mu$ representing the class given by $\mu \in \mathbf{C}^*$ by the familiar quotient construction (c.f. Section 2.1). The bundle $\mathcal{V}_\mu$ restricts to the bundle $\mathcal{L}_\mu$ on the Hopf surfaces X_p, where $\mathcal{L}_\mu$ is the bundle defined in Subsection 2.5.1. It is easy to check that

7.2.6. $$\overline{\tau^*\mathcal{V}_\mu} \cong \mathcal{V}_{\bar{\mu}}.$$

Since $H^2(Q;\mathcal{O}_Q^*) = 0$, by (2.2.8) we have the following:

Lemma 7.2.7. *There is an exact sequence*

$$0 \to \mathrm{Pic}(Q) \to \mathrm{Pic}^0(Z,Q) \to \mathrm{Map}(Q,T) \to 1.$$

For $(a,b) \in \mathbf{Z} \times \mathbf{Z}$, let $\mathcal{O}_Z(a,b)$ denote the bundle $\pi^*\mathcal{O}_Q(a,b)$. By (7.2.7) any line bundle on Z which is trivial on restriction to fibres of π is isomorphic to $\mathcal{O}_Z(a,b)$ for some $a,b \in \mathbf{Z}$. The relationships between the various bundles are described in the following lemma.

Lemma 7.2.8. *There exist flat holomorphic line bundles $\mathcal{U},\mathcal{V}$ on Z such that*

$$\mathcal{O}_Z(1) \cong \mathcal{O}_Z(1,0)\mathcal{U} \cong \mathcal{O}_Z(0,1)\mathcal{V},$$

The line bundles $\mathcal{U},\mathcal{V}$ satisfy

$$\mathcal{U}\mathcal{V} \cong \mathcal{O}_Z \text{ and } \mathcal{U}\mathcal{V}^{-1} \cong \mathcal{V}_\lambda.$$

Proof of 7.2.8: Let F be a twistor line, then we have

$$\mathcal{O}_Z(1)|_F \cong \mathcal{O}_Z(1,0)|F \cong \mathcal{O}_Z(0,1)|_F \cong \mathcal{O}_F(1).$$

By the version of the Ward correspondence for the group $\mathbf{C}^*$ [9], there are flat line bundles $\mathcal{U},\mathcal{V}$ on Z such that

$$\mathcal{O}_Z(1) \cong \mathcal{O}_Z(1,0)\mathcal{U} \cong \mathcal{O}_Z(0,1)\mathcal{V}.$$

For any $p \in \mathbf{P}^1_0$, the adjunction formula gives

$$\mathcal{K}_Z|_{X_p} \cong \mathcal{K}_{X_p} \otimes \mathcal{N}^*_{X_p} \cong \mathcal{K}_{X_p},$$

where $\mathcal{N}_{X_p}$ is the normal bundle in Z of the Hopf surface X_p. This bundle is trivial, since the fibration $\pi : Z \to Q$ is locally trivial. Hence, using the isomorphism $\mathcal{K}_Z \cong \mathcal{O}_Z(-4,0)\mathcal{U}^{-4}$, we have

$$\mathcal{U}^{-4}|_{X_p} \cong \mathcal{K}_{X_p} \cong \mathcal{V}_\lambda^{-2}|_{X_p},$$

and so $\mathcal{U}^4 \cong \mathcal{V}_\lambda^2$. The reality of $\mathcal{O}_Z(1)$ and $\mathcal{O}_Z(1,0)$ imply that $\mathcal{U}$ must be real, hence $\mathcal{U} \cong \mathcal{V}_{\pm\sqrt{\lambda}}$. Furthermore,

$$\mathcal{K}_Z \cong \pi^*\mathcal{K}_Q \cong \mathcal{O}_Z(-2,-2) \cong \mathcal{O}_Z(-2,-2)\mathcal{U}^2\mathcal{V}^2,$$

so $\mathcal{U}^2\mathcal{V}^2 \cong \mathcal{O}_Z$. We then deduce that $\mathcal{V} \cong \mathcal{V}^{-1}_{\pm\sqrt{\lambda}}$. Since $\mathcal{U}\mathcal{V}^{-1} \cong \mathcal{O}_Z(-1,-1)$, the bundle $\mathcal{U}\mathcal{V}^{-1}$ must be trivial on the fibres of π and so the pair must be related by $\mathcal{U}\mathcal{V}^{-1} \cong \mathcal{V}_\lambda$ and $\mathcal{U}\mathcal{V} \cong \mathcal{O}_Z$. $\boxed{7.2.8}$

Lemma (7.2.8) allows us to identify the terms in the exact sequence of (7.2.7). For any point $x \in Q$, (7.2.8) shows that $\mathcal{O}_Z(1)|_{\pi^{-1}(x)} \cong \mathcal{U}|_{\pi^{-1}(x)}$, so all line bundles on Z restrict to degree 0 line bundles on fibres of π, giving

$$\mathrm{Pic}^0(Z,Q) \cong \mathrm{Pic}(Z) \cong \mathbf{C}^* \times \mathbf{Z}.$$

Since all holomorphic maps from Q to T are constant and $\mathrm{Pic}(Q) \cong \mathbf{Z} \times \mathbf{Z}$, the exact sequence of (7.2.7) is seen to be:

$$0 \to \mathbf{Z} \times \mathbf{Z} \xrightarrow{\pi^*} \mathbf{C}^* \times \mathbf{Z} \xrightarrow{\rho} T \to 0,$$

where

$$\pi^*(a,b) = (u^{-a}v^{-b}, a+b)$$

and

$$\rho(\mu, n) = \langle \mu.u^n \rangle = \langle \mu.v^{-n} \rangle,$$

where u, v are the classes in $\mathbf{C}^*$ of the line bundles $\mathcal{U}, \mathcal{V}$ respectively.

7.3. Holomorphic SL(2, C) bundles on twistor space

Suppose $\mathcal{E} \to Z$ is a holomorphic $\mathrm{SL}(2,\mathbf{C})$ bundle such that for each $p \in \mathbf{P}^1_0$ and $q \in \mathbf{P}^1_\infty$, $c_2(\mathcal{E}|X_p) = c_2(\mathcal{E}|X'_q) = k > 0$. Using the description of Z as an elliptic 3-fold we may analyse $\mathcal{E}$ fibrewise as we did in Chapter 3 and Chapter 5. The fibration of Z by Hopf surfaces and (2.5.2) show that the bundle $\mathcal{E}$ must be of type (1) or (2) on restriction to a generic fibre of π. One can now repeat the divisor construction. Let $\mathcal{P} \to Z \times \mathrm{Pic}(Z)$ be a Poincaré line bundle and suppose $p_1 : Z \times \mathrm{Pic}(Z) \to Z$ is projection. We can then consider the direct image sheaf $R^1\bar{\pi}_*(\mathcal{P} \otimes p_1^*\mathcal{E})$ where

$$\bar{\pi} = \pi \times \mathrm{id} : Z \times \mathrm{Pic}(Z) \to Q \times \mathrm{Pic}(Z).$$

This is a torsion sheaf which by (2.2.16) is supported precisely on the divisor $\bar{D}$ consisting of those points $(x, \mathcal{L}) \in Q \times \mathrm{Pic}(Z)$ for which $h^1(\pi^{-1}(x); \mathcal{L}\mathcal{E}) \neq 0$, i.e., for which $\mathcal{L}|_{\pi^{-1}(x)}$ is a sub-bundle of $\mathcal{E}|_{\pi^{-1}(x)}$. The invariance of $\bar{D}$ under the actions of $\mathrm{Pic}(Q)$ and $\mathbf{Z}_2$ gives corresponding divisors Σ on $Q \times T$ and D on $Q \times \mathbf{P}^1$ respectively. Clearly the intersection of Σ and D with the divisors $\{p\} \times \mathbf{P}^1_\infty \times T$ and $\{p\} \times \mathbf{P}^1_\infty \times \mathbf{P}^1$ respectively gives the corresponding objects for the Hopf surface X_p, and similarly for the X'_q. Hence D is a $(k,k,1)$ divisor in $Q \times \mathbf{P}^1$ and the map $Q \times T \to Q \times \mathbf{P}^1$ induced by the hyperelliptic involution restricts to a double cover $\nu : \Sigma \to D$.

Let $\pi_Q : Q \times \mathbf{P}^1 \to Q$ be projection. Since $\mathcal{E}$ is generically of type (1) or (2) on the fibres of π, the fibres of π_Q intersect the divisor D in one point generically, defining a rational map $\alpha : Q \to \mathbf{P}^1$. Hence the divisor D reduces as

$$D = G_\alpha + \sum_{i=1}^{r} \pi_Q^*(C_i),$$

where G_α is the graph of α and $C = \{C_i : i = 1, \ldots, r\}$ is a collection of divisors on Q. The divisors C_i correspond to the fibres of π where $\mathcal{E}$ restricts to a bundle of type (3). Since on Hopf surfaces $\mathcal{E}$ is generically of types (1) or (2) the set C can contain no multiples of $(1,0)$ or $(0,1)$ curves, so each divisor $\pi^*(C_i)$ meets elements of the families of Hopf surfaces X_p and X'_q transversely.

Let $\rho : Q \times \mathbf{P}^1 \to \mathbf{P}^1$ and $\tilde{\rho} : Q \times T \to T$ be the projections onto the second factors. Denote by D_p, D_q and $D(\mu)$ the intersections of D with the quadric surfaces $\{p\} \times \mathbf{P}^1_\infty \times \mathbf{P}^1$, $\mathbf{P}^1_0 \times \{q\} \times \mathbf{P}^1$ and $Q \times \{\mu\}$ respectively. We can think of $\mathcal{E}|X_p$ as a family of deformations of an SL(2, C) bundle on a Hopf surface parameterised by $p \in \mathbf{P}^1_0$. The curves D_p are then the corresponding deformations of the divisor. Similarly the D'_q are deformations in the transverse direction. As we deform the bundle, its divisor and fibres where it is of type (3) may change. On certain exceptional Hopf surfaces, as we shall see, the degree of the graph part of the divisor may jump and new fibres of type (3) appear.

The divisors $D(\mu)$ play a different rôle and have a nice geometric interpretation. Here there are two possibilities: either $D(\mu) = Q \times \{\mu\}$ for some $\mu \in \mathbf{C}^*$, or the projection $\pi_{Q*}(D(\mu))$ is a (k,k) divisor on Q for all $\mu \in T$.

In the first case, the rational map $\alpha : Q \to \mathbf{P}^1$ is constant. The bundle $\mathcal{E}$ can then be written as an extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}\mathcal{J}_Y \to 0,$$

as in Chapter 3, where $\mathcal{L}$ is a line bundle on Z and Y is codimension two locally complete intersection subscheme.

In the second case, where α is non-constant, the curves $D(\mu)$ define a pencil in the linear system $|\mathcal{O}_Q(k,k)|$ on Q. A point $x \in Q$ lies on an element $D(\mu)$ if and only if $\mathcal{E}|_{\pi^{-1}(x)}$ admits $\mathcal{L}_\mu|_{\pi^{-1}(x)}$ as a sub-bundle, where $\mathcal{L}_\mu$ is any lift of $\mu \in \mathbf{P}^1 \cong \mathrm{Pic}(Z)/(\mathrm{Pic}(Q) \times \mathbf{Z})$ to $\mathrm{Pic}(Z)$. The base locus of the pencil consists of the fixed components C_i and points of Q where the rational map α is not well-defined. In terms of the bundle $\mathcal{E}$, the base locus consists precisely of those points of Q over which $\mathcal{E}$ is of type (3).

The two points of view are entirely equivalent. Given a pencil $D(\mu)$ of (k,k) curves on Q we may remove the fixed components $\{C_i : i = 1, \ldots, r\}$ to obtain a pencil of curves whose base locus is of codimension at least 2. The blow-up of Q in the base locus is a divisor G_α on $Q \times \mathbf{P}^1$ which is the graph of a rational map $\alpha : Q \to \mathbf{P}^1$. The pencil $D(\mu)$ then lifts naturally to a *base-point-free* pencil on G_α. Note that above a point b contained in the base locus, the surface G_α contains an exceptional divisor $E = \pi_Q^{-1}(b) \cong \mathbf{P}^1$ with $E^2 = -1$.

Adding in the contribution from the fixed components gives a $(k,k,1)$ divisor

$$G_\alpha + \sum_{i=1}^{r} \pi_Q^*(C_i)$$

on $Q \times \mathbf{P}^1$. See [30, p.490] for further details.

7.4. Instantons, monopoles and spectral surfaces

Now suppose that F is an SU(2) bundle over $S^1 \times S^3$ with $c_2(F) = k > 0$ and equipped with an anti-self-dual connection A. Via the Ward correspondence A defines a holomorphic $\mathrm{SL}(2, \mathbf{C})$ structure on the pull-back $E = P^*F$ of F to Z. Since each element of the families of Hopf surfaces X_p, X'_q is diffeomorphic via P with $S^1 \times S^3$, we have $c_2(\mathcal{E}|X_p) = c_2(\mathcal{E}|X'_q) = k$. The results of Section 7.3 therefore apply and we can associate to $\mathcal{E}$ a surface $\Sigma \subset Q \times T$ which is a double cover of its divisor $D \subset Q \times \mathbf{P}^1$.

The divisor D is an element of the linear system $|\mathcal{O}(k, k, 1)|$, whose complex dimension is $2(k+1)^2 - 1$; i.e, quadratic in k. However, we know that $\dim_{\mathbf{C}} \mathcal{M}_k = 4k$ is linear in k, so D must be constrained in some way by the conditions on $\mathcal{E}$ required for its descent to $S^1 \times S^3$. The first constraint is given by the reality conditions. Using the reality of $\mathcal{O}_Z(1)$ and (7.2.6) we see that the real structure τ_p on $\mathrm{Pic}(Z) \cong \mathbf{C}^* \times \mathbf{Z}$ is given by

$$\tau_p(\mu, n) = (\overline{\mu}, n).$$

The anti-holomorphic involution τ_p commutes with the actions of $\mathrm{Pic}(Q)$ and $\mathbf{Z}_2$ on $\mathrm{Pic}(Z)$ and so descends to give real structures on both T and $\mathbf{P}^1$.

Suppose now that for a given point $x \in Q$ and line bundle $\mathcal{L}$ on Z we have $h^0(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) \neq 0$. Applying the real structure and using the fact that $\overline{\tau^*\mathcal{E}} \cong \mathcal{E}$ gives

$$h^0(\pi^{-1}(\tau(x)); (\overline{\tau^*\mathcal{L}})^{-1}\mathcal{E}) \neq 0.$$

This then proves:

Lemma 7.4.1. *The surfaces Σ and D are invariant under the real structures on $Q \times T$ and $Q \times \mathbf{P}^1$.*

Definition 7.4.2. If $\mathcal{E} \to Z$ is an instanton bundle, we call the associated real surface $\Sigma \subset Q \times T$ the *spectral surface* of $\mathcal{E}$.

We now attempt to justify this definition and outline the relationship with monopoles. We shall not to go into details, but refer the reader to the cited papers. First let us recall Hitchin's definition of the spectral curve of an SU(2) monopole on $\mathbf{R}^3$ [7, 33]. Let E be the standard product bundle $\mathbf{R}^3 \times \mathbf{C}^2$ over $\mathbf{R}^3$ with its usual SU(2) structure and let $\mathrm{sl}(E)$ denote the real rank 3 vector bundle corresponding to skew-adjoint endomorphisms of E. A monopole on E consists of a pair (∇, Φ), where ∇ is an SU(2) connection on E and Φ is a section of $\mathrm{sl}(E)$, which satisfy the *Bogomolnyi equation*

7.4.3. $$F_\nabla = - * \nabla\Phi,$$

along with further assumptions on the asymptotic behaviour of ∇ and Φ (see [33] for details). The mini-twistor space $\mathbf{T}$ of $\mathbf{R}^3$ is the space of oriented geodesics in $\mathbf{R}^3$, and can be constructed as follows. The space $\mathbf{R}^3$ is the quotient of $\mathbf{R}^4$ by the group $\mathbf{R}$ acting by translations. This action lifts to a holomorphic action on the twistor space $Z(\mathbf{R}^4)$ of $\mathbf{R}^4$, where it complexifies to give an action of $\mathbf{C}$. The quotient $\mathbf{T} = Z(\mathbf{R}^4)/\mathbf{C}$ is isomorphic to $T\mathbf{P}^1$, the tangent bundle of $\mathbf{P}^1$. For each oriented geodesic γ parameterised by path length s, Hitchin considers the *scattering operator*

$$D_\gamma : \Gamma(E|_\gamma) \to \Gamma(E|_\gamma),$$

given by

7.4.4. $$D_\gamma\sigma = (\nabla_{\partial/\partial s} - i\Phi)\sigma.$$

Those points $\gamma \in \mathbf{T}$ for which (7.4.4) has a non-trivial solution which decays exponentially as $s \to \pm\infty$ define a compact complex curve $S \subset \mathbf{T}$: the *spectral curve* of the monopole.

We now show how our surface Σ fits into this picture. We can interpret instantons on $S^1 \times S^3$ as monopoles on an infinite-dimensional vector bundle over S^3 associated to an 'extended' loop-group. This approach was first considered by Garland and Murray [27, 28], who applied it to instantons on $S^1 \times \mathbf{R}^3$ and $S^1 \times H^3$ (where H^3 denotes hyperbolic 3-space).

For $\mu_0 \in \mathbf{R}^*$, let $S^1_{\mu_0}$ denote the quotient $\mathbf{R}/(2\pi/\mu_0)$ with the metric inherited from the usual metric on $\mathbf{R}$. We reserve the notation S^1 for the case $\mu_0 = 1$, when the metric is the standard one. The standard linear coordinate on $\mathbf{R}$ induces coordinates t and θ on S^1_μ and S^1 respectively such that the 1-forms dt and $d\theta$ are of unit length with respect to induced metrics. We express S^3 as the union $D_0 \cup D_\infty$, where D_0 and D_∞ are both diffeomorphic to an open 3-disc $D \subset \mathbf{R}^3$, and $D_0 \cap D_\infty \cong \mathbf{R} \times S^2$. Any SU(2) bundle over $S^1 \times D$ is trivial. Hence, given any SU(2) bundle E over $S^1_{\mu_0} \times S^3$ there is a map $g : S^1_{\mu_0} \times (D_0 \cap D_\infty) \to \mathrm{SU}(2)$ such that E is isomorphic to the quotient bundle

7.4.5. $$(S^1_{\mu_0} \times D_0 \times \mathbf{C}^2 \sqcup S^1_{\mu_0} \times D_\infty \times \mathbf{C}^2)/\sim$$

given by identifying the points (t, x, ξ) and $(t, x, g(t, x)\xi)$, whenever $(t, x) \in S^1_{\mu_0} \times (D_0 \cap D_\infty)$. If ∇ is an SU(2) connection on E, then with respect to the natural local gauges given by (7.4.5) ∇ defines gauge potentials A_0 and A_∞, and corresponding to the product structure of $S^1_{\mu_0} \times S^3$ there are unique decompositions

$$A_0 = \alpha_0 + \phi_0 dt;$$
$$A_\infty = \alpha_\infty + \phi_\infty dt,$$

where $\alpha_0 \in \Omega^1(D_0, \mathfrak{su}(2))$ and $\phi_0 \in \Omega^0(D_0, \mathfrak{su}(2))$ (with analogous definitions for α_∞ and ϕ_∞). These gauge potentials are related by the gauge transformation given by

$$A_\infty = g^* A_0,$$

i.e.,

$$\alpha_\infty = g^{-1}d_3 g + g^{-1}\alpha_0 g,$$
$$\phi_\infty = g^{-1}\frac{\partial g}{\partial t} + g^{-1}\phi_0 g, \qquad 7.4.6.$$

where d_3 denotes exterior differentiation on the 3-manifold $D_0 \cap D_\infty$. Now let $\Omega SU(2)$ denote the *unbased* loop group of SU(2), and denote its Lie algebra, the space of smooth maps from S^1 to $\mathbf{su}(2)$, by $L\Omega SU(2)$. Let $\tilde{\Omega}SU(2)$ denote the semi-direct product $S^1 \ltimes \Omega SU(2)$ with multiplication given by

7.4.7. $$(\nu, f(\theta)).(\varphi, h(\theta)) = (\nu + \varphi, f(\theta)h(\theta + \nu)).$$

The Lie algebra of $\tilde{\Omega}SU(2)$ is then the direct sum

$$L\tilde{\Omega}SU(2) = L\Omega SU(2) \oplus i\mathbf{R}D_\theta.$$

and Lie bracket given by

$$[X + ixD_\theta, Y + iyD_\theta] \stackrel{\text{def}}{=} [X, Y] + x\frac{\partial Y}{\partial \theta} - y\frac{\partial X}{\partial \theta},$$

where $X, Y \in L\Omega SU(2)$ and $x, y \in \mathbf{R}$. Using (7.4.7) one easily verifies that if $\tilde{g} = (\nu, g) \in \tilde{\Omega}SU(2)$ and $\Phi = \phi + i\mu D_\theta \in L\tilde{\Omega}SU(2)$ then

7.4.8. $$(\tilde{g}^{-1}\Phi\tilde{g})(\theta) = (g^{-1}\phi g + \mu g^{-1}\partial g/\partial\theta)(\theta - \nu) + i\mu D_\theta.$$

The map $t \mapsto \mu_0 t$ identifies $S^1_{\mu_0}$ with S^1 and the equations (7.4.6) become

$$\phi_\infty + i\mu_0 D_\theta = \tilde{g}^{-1}(\phi_0 + i\mu_0 D_\theta)\tilde{g},$$
$$\alpha_\infty = \tilde{g}^{-1}d_3\tilde{g} + \tilde{g}^{-1}\alpha_0\tilde{g},$$

where now $\tilde{g} = (1, g)$ is regarded as a map $\tilde{g} : D_0 \cap D_\infty \to \tilde{\Omega}SU(2)$. Hence, setting

$$\Phi_\nu = \phi_\nu + i\mu_0 D_\theta;$$
$$a_\nu = \alpha_\nu,$$

for $\nu = 0, \infty$, then the pairs (a_ν, Φ_ν) transform in the correct way with respect to the clutching function

$$\tilde{g} : D_0 \cap D_\infty \to \tilde{\Omega}SU(2),$$

to define a connection and Higgs field pair (a, Φ) on the corresponding $\tilde{\Omega}SU(2)$ bundle over S^3. If the connection ∇ on E is anti-self-dual then the pair (a, Φ) satisfies the Bogomolnyi equation

7.4.9. $$* F_a = -d_a\Phi,$$

where d_a denotes the covariant derivative induced by the connection a.

Our Hopf surface X defined by $\lambda \in \mathbf{R}_{>1}$ is isometric to $S^1_{\mu_0} \times S^3$, where $\mu_0 = 2\pi/\log\lambda$. Hence we may regard solutions to the anti-self-duality equations on X as monopoles on the bundle $\tilde{E}$, with fibre the set of smooth maps $S^1 \to \mathbf{C}$, associated to the principal $\tilde{\Omega}\mathrm{SU}(2)$ bundle defined above. Now S^3 is the quotient of $S^1_{\mu_0} \times S^3$ by the obvious S^1-action, which lifts to Z and complexifies to give the $\mathbf{C}^*$-action defined by (7.2.3). The quotient of Z by $\mathbf{C}^*$ is Q, the mini-twistor space of S^3. Points of Q parameterise oriented great circles in S^3. Following the approach of Hitchin, for $\mu \in \mathbf{C}$ and $\gamma \in Q$ we define a *twisted scattering operator* $D_{\gamma,\mu}$ for an $\tilde{\Omega}\mathrm{SU}(2)$ monopole (a, Φ) over S^3 by

7.4.10.
$$D_{\gamma,\mu}\sigma = \nabla^a_{\partial/\partial s}\sigma - i(\Phi\sigma + \mu\sigma),$$

where s is path-length on the oriented geodesic γ and σ is a section of $\tilde{E}$ restricted to γ. We define a subset $\tilde{\Sigma} \subset Q \times \mathbf{C}$ to be the set of all points (γ, μ) such that the operator $D_{\gamma,\mu}$ has non-trivial kernel. It is easy to check that if σ is a solution to $D_{\gamma,\mu}\sigma = 0$, then $e^{2\pi i s}\sigma$ and $e^{2\pi i \theta}\sigma$ are solutions to $D_{\gamma,\nu}$ with $\nu = \mu + 2\pi$ and $\nu = \mu - 4\pi^2 i/\log\lambda$ respectively. Hence $\tilde{\Sigma}$ is invariant under the equivalence relation $\sim$ on $Q \times \mathbf{C}$ generated by

$$(\gamma, \mu) \sim (\gamma, \mu + 2\pi) \sim (\gamma, \mu - 4\pi^2 i/\log\lambda),$$

and so descends to a subset Σ' in the quotient space $Q \times T$, where

$$T = \mathbf{C}/\{2\pi, -4\pi^2 i/\log\lambda\} \cong \mathbf{C}^*/\langle\lambda\rangle$$

is the same elliptic curve we have used throughout.

Of course Σ' is precisely the spectral surface Σ defined above. The operator $D_{\gamma,0}$ can be identified with the restriction to $\pi^{-1}(\gamma) \subset Z$ of the $\bar{\partial}$-operator of the instanton bundle $\mathcal{E}$ on Z corresponding to the anti-self-dual connection A. The operator $D_{\gamma,\mu}$ is the corresponding $\bar{\partial}$-operator associated with the twisted bundle $\mathcal{E}\mathcal{V}_\mu$, where $\mathcal{V}_\mu$ is the flat $\mathbf{C}^*$-bundle on X defined earlier, and has non-zero kernel precisely when on $F = \pi^{-1}(\gamma)$ the bundle $\mathcal{E}|_F$ admits $\mathcal{V}_\mu|_F$ as a sub-bundle.

Let us now return to consider the divisor of a k-instanton bundle on Z. When the graph part of the divisor is non-constant, the reality condition means that the corresponding pencil of (k, k) curves is real; i.e, $\tau'(D(\mu)) = D(\tau_p(\mu))$ for all $\mu \in \mathbf{P}^1$. The reality conditions constrain the divisor to be an element of a $2(k+1)^2 - 1$ *real* dimensional family of curves, which is still quadratic in k. There must therefore be further non-trivial constraints arising from the remaining descent conditions. The effect of these conditions on D is not clear. There are topological constraints on the pencil relating the Euler characteristic of Q to that of the Euler characteristics of the generic and exceptional elements of the pencil [30, p.509]. This reduces the problem to one of enumerative geometry and examining the various possibilities for singularities and exceptional elements of the pencil. Even for small values of k, however, the situation becomes very complicated. In the simplest case where $k = 1$, however, the additional conditions appear to be non-effective and we are able to give describe explicitly how all 1-instanton bundles arise. In the sequel we shall therefore concern ourselves exclusively with the case $k = 1$. First we describe the two possible types of divisor.

The divisor is an element of the linear system $|\mathcal{O}(1,1,1)|$ and is given by an equation

$$Z_0((X_0, X_1)A_1(Y_0, Y_1)^T) - Z_1((X_0, X_1)A_0(Y_0, Y_1)^T) = 0,$$

where $[X_0, X_1], [Y_0, Y_1]$ and $[Z_0, Z_1]$ are homogeneous coordinates on $\mathbf{P}^1_0, \mathbf{P}^1_\infty$ and $\mathbf{P}^1$ respectively, and A_0, A_1 are 2×2 complex matrices, not both zero. The corresponding rational map $\alpha : Q \to \mathbf{P}^1$ is given by

$$\alpha([X],[Y]) = [XA_0Y^T, XA_1Y^T],$$

where we write X for the vector (X_0, X_1) and similarly for Y. There are two distinct cases. If there exists a point $(\mu_0, \mu_1) \in \mathbf{C}^*$ such that

$$\mu_1 A_0 - \mu_0 A_1 = 0,$$

then the map α is constant with value $[\mu_0, \mu_1]$. The divisor then reduces as the sum of the graph of α and $\pi^{-1}(C)$ where C is the real $(1,1)$ divisor given by $XA_0Y^T = 0$ or $XA_1Y^T = 0$. In this case the divisors D_p and D'_q are singular for all $p \in \mathbf{P}^1_0$ and $q \in \mathbf{P}^1_\infty$.

Otherwise the map α is well-defined only outside the two points of intersection b_1, b_2 of the two $(1,1)$ curves given by $XA_iY^T = 0$ for $i = 1,2$. The reality condition gives $b_1 = \tau(b_2)$. The divisor D is the graph of α and it is easily checked that it is a smooth surface, being the blow-up of Q in the points b_1 and b_2. In the next two sections we shall consider each of the two cases in turn.

7.5. The Atiyah-Ward construction for 1-instantons

Suppose $\mathcal{E} \to Z$ is an instanton bundle, with instanton number $k = 1$, which can be constructed by the Atiyah-Ward construction. There then exists a line bundle $\mathcal{L}$ on Z and a 1-dimensional locally complete intersection subscheme Y of Z such that $\mathcal{E}$ can be written as an extension

7.5.1. $$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}\mathcal{J}_Y \to 0.$$

Let $s \in H^0(\mathcal{L}^{-1}\mathcal{E})$ be a non-trivial section which defines (7.5.1). As in Section 3.2 this shows that the graph of $\mathcal{E}$ is that of a constant map α whose value everywhere is given by the equivalence class of $\mathcal{L}$ in $\mathbf{P}^1$. More generally, if the graph of $\mathcal{E}$ is constant, then the spectral surface Σ decomposes

7.5.2. $$\Sigma = Q \times \{\mu_1\} + Q \times \{\mu_2\} + \pi_Q^* C,$$

for some $\mu_1, \mu_2 \in T$ with $\mu_2 = \iota(\mu_1)$ and (k,k) curve C on Q. We then have the following analogue of (2.4.2).

Lemma 7.5.3. *Let $\mathcal{E}$ be a k-instanton bundle on Z and suppose the spectral surface $\Sigma \subset Q \times T$ associated to $\mathcal{E}$ decomposes as in (7.5.2). If $\mathcal{E}$ does not restrict to a direct sum of half-periods on a generic fibre of π, then there exist two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(Z)$ such that any line bundle admitting a non-trivial map to $\mathcal{E}$ is isomorphic to one in the set*

$$\{\mathcal{L}_1 \otimes \pi^*\mathcal{O}_Z(-a,-b), \mathcal{L}_2 \otimes \pi^*\mathcal{O}_B(-a,-b) \in \mathrm{Pic}(Z) : a, b \geq 0\}.$$

The line bundles satisfy $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \pi_*(\mathcal{L}_2^{-1}\mathcal{E}) \cong \mathcal{O}_Q$. If $\mathcal{E}$ is of type (2) on a generic fibre then $\mathcal{L}_1 \cong \mathcal{L}_2$.

Proof of 7.5.3: Let $\mathcal{L}'$ be a line bundle on Z with graph $Q \times \{\mu_2\}$, then the first direct image sheaf $R^1\pi_*(\mathcal{L}'\mathcal{E})$ has rank 1. By relative Serre duality [36] there is an isomorphism $\pi_*((\mathcal{L}')^{-1}\mathcal{E}) \cong (R^1\pi_*\mathcal{L}'\mathcal{E})^*$. For any coherent sheaf S on Q, its dual S^* is reflexive and hence locally free outside a codimension 3 subset of Q, i.e., everywhere. Hence $\pi_*((\mathcal{L}')^{-1}\mathcal{E})$ is locally free. The proof now proceeds exactly as in (2.4.2). $\boxed{7.5.3}$

Returning to the case when $k = 1$, by (3.2.2), the bundle $\mathcal{E}$ satisfies the hypotheses of (7.5.3) and has $\mathcal{L}$ as a maximal destabilising line bundle. Applying $\overline{\tau^*(\cdot)}$ to the exact sequence 7.5.1 and using the reality of $\mathcal{E}$ gives the exact sequence

7.5.4. $$0 \to \overline{\tau^*\mathcal{L}} \to \mathcal{E} \to (\overline{\tau^*\mathcal{L}})^{-1}\mathcal{J}_{\tau(Y)} \to 0.$$

Hence $\overline{\tau^*\mathcal{L}}$ is also a maximal destabilising line bundle for $\mathcal{E}$.

Lemma 7.5.5. *If $\mathcal{E} \to Z$ is a 1-instanton bundle with maximal destabilising line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ then*

$$\mathcal{L}_1\mathcal{L}_2 \cong \mathcal{O}_Z(-1,-1).$$

Proof of 7.5.5: By definition $\pi_*(\mathcal{L}_1^{-1}\mathcal{E}) \cong \mathcal{O}_Q$. Since $h^1(\pi^{-1}(x); \mathcal{L}^{-1}\mathcal{E}) = 1$ for all $x \in Q$, the sheaf $R^1\pi_*(\mathcal{L}_1^{-1}\mathcal{E})$ is a line bundle on Q by Grauert's direct image theorem, and hence isomorphic to $\mathcal{O}_Z(a,b)$ for some $a, b \in \mathbf{Z}$. By relative Serre duality,

$$\pi_*(\mathcal{L}_1\mathcal{E}) \cong \mathcal{O}_Q(-a,-b),$$

and so

$$\pi_*((\mathcal{L}_1^{-1} \otimes \mathcal{O}_Z(-a,-b))^{-1}\mathcal{E}) \cong \mathcal{O}_Q,$$

giving $\mathcal{L}_2 \cong \mathcal{L}_1^{-1} \otimes \mathcal{O}_Z(-a,-b)$ by uniqueness. Restricting to generic elements of the families of Hopf surfaces X_p, X'_q and using (3.4.12) gives $a = b = -1$.

$\boxed{7.5.5}$

The curve Y may be written as

$$Y = Y_0 + \sum_{i=1}^{r} T_i,$$

where for $i = 1, \ldots, r$ the curve $T_i = \pi^{-1}(x_i)$ for some point $x_i \in Q$ and Y_0 is a (possibly non-reduced) curve with compact support such that $C_0 = \pi_*(Y_0)$ is a divisor on Q and $\pi|_{Y_0} : Y_0 \to C_0$ is a map of finite degree.

Restricting (7.5.1) to a Hopf surface X_p, for some $p \in \mathbf{P}^1_0$, gives an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E}|_{X_p} \to (\mathcal{L}')^{-1}\mathcal{J}_{Y_p} \to 0,$$

for some line bundle $\mathcal{L}'$ on X_p, where Y_p is the (0-dimensional) zero-scheme of the induced section $s|X_p$ of $\mathcal{L}^{-1}\mathcal{E}|_{X_p}$. Consequently, by the results of Chapter 3 we have $l(Y_p) = 1$ and $\mathcal{E}$ has a fibre of type (3) at $\pi(Y_p)$. This also shows that C_0 is a smooth $(1,1)$ divisor on Q and Y_0 is a section of the trivial elliptic fibration $Z|_{C_0}$. (Recall that Z is a quotient of $\mathcal{O}(1,-1)$, which is trivial on restriction to the complement of a $(1,1)$ curve on Q). Since the set of fibres of Z where $\mathcal{E}$ is of type (3) is preserved by the real structure, we see that the curve C_0 is real. The bundle $\mathcal{L}$ and curve Y are related by the constraint

7.5.6. $$\det \mathcal{N}_Y \cong \mathcal{L}^{-2}|_Y.$$

Since $\pi : Z \to Q$ is locally trivial, the normal bundles of the fibres $\mathcal{N}_{T_i}$ are all trivial (for $i = 1, \ldots, r$). Hence (7.5.6) becomes

$$\mathcal{L}^{-2}|_{Y_0} \cong \det \mathcal{N}_{Y_0}.$$

The following lemma follows from [32, p.292].

Lemma 7.5.7. *There is an isomorphism*

$$\mathrm{Pic}(\mathbf{P}^1 \times T) \cong \mathrm{Pic}(\mathbf{P}^1) \times \mathrm{Pic}(T).$$

Furthermore, given a line bundle $\mathcal{L}$ on $\mathbf{P}^1 \times T$, the restrictions of $\mathcal{L}$ to $\mathbf{P}^1 \times \{t\} \cong \mathbf{P}^1$ are isomorphic for all $t \in T$.

The line bundles $\mathcal{L}$ which satisfy (7.5.6) are determined by the following results.

Lemma 7.5.8. *If C_0 is a smooth real $(1,1)$ curve on Q then the trivial bundle $Z|_{C_0}$ admits a twistor line as a section.*

Proof of 7.5.8: Suppose C_0 is the graph of an automorphism $\Phi : \mathbf{P}^1_\infty \to \mathbf{P}^1_0$ given by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{C}).$$

If C_0 is real, then $b = -c$ and $d = -a$. The real line

$$P^{-1}(a + cj) = \{[az_2 - cz_3, cz_2 + az_3, z_2, z_3] \in Z : (z_2, z_3) \in \mathbf{C}^2 \setminus \{0\}\}$$

then defines a section of $Z|_{C_0}$. $\boxed{7.5.8}$

Corollary 7.5.9. *The line bundle $\mathcal{L}$ of (7.5.1) satisfies $\mathcal{L}|_{Y_0} \cong \mathcal{O}_{Y_0}(-1)$.*

Proof of 7.5.9: By (7.5.7) and (7.5.8), $\deg(\det \mathcal{N}_{Y_0}) = \deg(\det \mathcal{N}_F)$, where F is any twistor line. Since $\mathcal{N}_F \cong \mathcal{O}_F(1) \oplus \mathcal{O}_F(1)$, we have $\mathcal{L}^{-2}|_{Y_0} \cong \mathcal{O}_{Y_0}(2)$, giving the result. $\boxed{7.5.9}$

As a result of (7.5.9), we must have $\mathcal{L} \cong \mathcal{V}_\mu(-1)$ for some $\mu \in \mathbf{C}^*$, and $\overline{\tau^*\mathcal{L}} \cong \mathcal{V}_{\bar\mu}(-1)$. The map τ either fixes or interchanges maximal destabilising line bundles. Accordingly there are two cases to consider:

(i) $\mathcal{L} \cong \overline{\tau^*\mathcal{L}}$;

(ii) $\mathcal{L} \cong (\overline{\tau^*\mathcal{L}})^{-1} \otimes \mathcal{O}_Z(-1,-1)$.

In the first case we obtain $\mu = \bar\mu$, so $\mathcal{V}_\mu$ is the pull-back of a flat $\mathbf{R}^*$-bundle on $S^1 \times S^3$, and $\tau(Y) = Y$. Consequently r must be even, $r = 2m$ say, and

7.5.10. $$Y = Y_0 + \sum_{i=1}^{m}(T_i + \tau(T_i))$$

with Y_0 a twistor line.

Suppose now that F is a twistor line passing through T_1, and hence also through $\tau(T_1)$. Restricting the exact sequence (7.5.1) to F and using (7.5.9) gives

$$0 \to \mathcal{O}_F(n) \to \mathcal{E}|_F \to \mathcal{O}_F(-n) \to 0,$$

where $n \geq 1$. Since $H^1(\mathcal{O}_F(2n)) = 0$ for $n \geq 1$, the sequence must split contradicting the triviality of $\mathcal{E}|_F$. Hence the integer m must be 0 and Y consists only of a twistor line. These are therefore the 't Hooft solutions.

The other maximal destabilising line bundle $\mathcal{L}'$ is given by $\mathcal{L}' = \mathcal{L}^{-1}\mathcal{O}_Z(-1,-1)$, and there is an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E} \to (\mathcal{L}')^{-1}\mathcal{J}_{Y'} \to 0,$$

where Y' is another twistor line, related to Y by

$$Y' = Y.(\mu^2)$$

in exactly the same way as in Section 3.7

Extensions of the form (7.5.1) are classified by $\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Y, \mathcal{L})$. If $(\mu^2) \neq 1$, then in the lower term sequence of the Leray spectral sequence

7.5.11. $$0 \to H^1(\pi_*\mathcal{L}^2) \to H^1(\mathcal{L}^2) \to H^0(R^1\pi_*\mathcal{L}^2) \to 0,$$

the first and last terms both vanish giving $H^1(\mathcal{L}^2) = 0$. Similarly $H^2(\mathcal{L}^2) \cong H^1(\mathcal{L}^{-2}\mathcal{K}_Z)$ vanishes, so in this case the fundamental exact sequence (3.3.5) gives

7.5.12. $$\mathrm{Ext}^1(\mathcal{L}^{-1}\mathcal{J}_Y, \mathcal{L}) \cong H^0(\mathcal{O}_Y) \cong \mathbf{C},$$

and choosing any generator for $H^0(\mathcal{O}_Y)$ gives a bundle $\mathcal{E}$, unique up to isomorphism.

If $(\mu^2) = 1$, then $\mathcal{L}^2 \cong \mathcal{O}_Z(-1,-1)$ and $\pi_*\mathcal{L}^2 \cong R^1\pi_*\mathcal{L}^2 \cong \mathcal{O}_Z(-1,-1)$. Using (7.5.11) and the isomorphism $\mathcal{K}_Z \cong \mathcal{O}_Z(-2,-2)$, we again obtain the isomorphism (7.5.12). Hence the isomorphism class of $\mathcal{E}$ depends only on the isomorphism class of $\mathcal{L}$ and the twistor line Y. Counting parameters, we see that the moduli space of such bundles has real dimension 5.

Remark 7.5.13. The connections corresponding to these solutions were obtained explicitly by Braam in [14]. There he gives a 3-parameter family of solutions given by twistor lines corresponding to points in a fixed fibre of the principal $S^1 \times S^1$ fibration

$$\pi : S^1 \times S^3 \to S^2$$

which is the smooth version of the Hopf surface fibration. Our two extra parameters correspond to the choice of this fibre.

In the second case where $\overline{\tau^*\mathcal{L}} \cong \mathcal{L}^{-1} \otimes \mathcal{O}_Z(-1,-1)$ is the second maximal destabilising line bundle, we obtain $\bar{\mu} = \mu^{-1}$, so V_μ is the pull-back of a flat U(1) bundle on $S^1 \times S^3$. The curves Y and $\tau(Y)$ are related by

7.5.14. $$\tau(Y) = Y.\langle \mu^2 \rangle,$$

giving $\tau(\pi(Y)) = \pi(Y)$. Hence Y must again be of the form

$$Y = Y_0 + \sum_{i=1}^{m}(T_i + \tau(T_i))$$

where Y_0 is a section of the trivial elliptic fibration $Z|_{C_0}$ for the smooth real (1,1) curve $C_0 = \pi(Y_0)$. As in the previous case, triviality on twistor lines shows $Y = Y_0$, locally free extensions $\mathcal{E}$ of the form (7.5.1) always exist and are unique up to isomorphism. The smooth real (1,1) curves C on Q form a 3 dimensional real variety. For each such C there is a 2-dimensional family of holomorphic sections Y of $Z|_C$, and for each choice of Y the condition (7.5.14) gives a discrete family of choices for $\mathcal{L}$. Counting parameters once more, the number of moduli of such bundles is again 5. Since $\dim_{\mathbb{R}} \mathcal{M}_1 = 8$, we obtain:

Proposition 7.5.15. *The generic element of the moduli space $\mathcal{M}_1$ cannot be constructed by the Atiyah-Ward construction.*

7.6. A construction of generic 1-instantons

In the previous section we saw that the Atiyah-Ward construction supplied only a 5-dimensional family of solutions in the moduli space $\mathcal{M}_1$. The generic 1-instanton must therefore have a divisor D which is the graph of a non-constant rational map from Q to $\mathbf{P}^1$. As we saw in Section 7.4, the surface D is the blow-up of Q in two points b and $\tau(b)$ which form the base locus of the associated pencil. This pencil lifts to a base-point-free pencil on D giving a fibration $\rho : D \to \mathbf{P}^1$ with sections given by the exceptional fibres $E_1 = \sigma^{-1}(b), E_2 = \sigma^{-1}(\tau(b))$ of the blow-up $\sigma : D \to Q$. Suppose $b = (p_0, q_0) \in \mathbf{P}^1_0 \times \mathbf{P}^1_\infty$. Let $\widehat{\pi}_0 : D \to \mathbf{P}^1_0$ and $\widehat{\pi}_\infty : D \to \mathbf{P}^1_\infty$ be the maps given by composing σ with π_0 and π_∞. The generic fibre of $\widehat{\pi}_0$ is a smooth rational curve, but there are two exceptional fibres corresponding to the points p_0 and $\tau(p_0) \in \mathbf{P}^1_0$. Similarly for $\widehat{\pi}_\infty$.

The fibres of $\rho : D \to \mathbf{P}^1$ define the pencil of $(1,1)$ curves on Q. Since the divisor D_{p_0} is singular, there exists a line bundle $\mathcal{V}_\mu$ on Z, for some $\mu \in \mathbf{C}^*$, such that $\mathcal{V}_\mu | X_{p_0}$ is a sub-bundle of $\mathcal{E}|X_{p_0}$. This means that the $(0,1)$ curve $\{p_0\} \times \mathbf{P}^1_\infty$ must be a component of the element $D(\mu)$ of the pencil. The other component must then be the $(0,1)$ curve $\mathbf{P}^1_0 \times \{\tau(q_0)\}$. Hence

$$D(\mu) = \{p_0\} \times \mathbf{P}^1_0 + \mathbf{P}^1_0 \times \{\tau(q_0)\}$$

is an exceptional element of the pencil. Since the pencil is real, $\tau'(D(\mu)) = D(\tau_p(\mu))$ is another exceptional element. Note that this means that $\tau_p(\mu) \neq \mu$, i.e., μ is non-real and in particular its image in T is not a half-period.

The situation is completely symmetric. The exceptional elements of the divisor correspond to the exceptional fibres of $\rho : D \to \mathbf{P}^1$, whose generic element is a smooth rational curve. Each component of $\rho^{-1}(\mu)$ and $\rho^{-1}(\tau(\mu))$ is a -1 curve on D. Hence we may may blow-down one component in each curve (the component in the fibres of $\widehat{\pi}_0$, say). Again by symmetry, the resulting surface is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Denote this blow-up by $\sigma' : D \to \mathbf{P}^1 \times \mathbf{P}^1$.

Let B denote the $(4,4)$ curve whose irreducible components are the four smooth $(1,1)$ curves in the pencil on Q corresponding to the fibres of π where $\mathcal{E}$ admits a half-period as a sub-bundle. The spectral surface Σ is the double cover of D with branch locus $\overline{B}$, the proper transform of B in D.

Moreover, since the exceptional fibres $\rho^{-1}(\mu)$ and $\rho^{-1}(\tau(\mu))$ of $\rho : D \to \mathbf{P}^1$ do not intersect $\overline{B}$, the corresponding fibration $\overline{\rho} : \Sigma \to T$ has four exceptional fibres, each of which is the sum of two -1 curves. Blowing down one -1 curve in each such fibre gives a ruled surface over T. Let $\{t_1, \ldots, t_4\} = q^{-1}\{\mu, \tau(\mu)\} \subset T$. Since these blow-downs take place away from the ramification divisor of the double cover, the two processes commute and we obtain:

Lemma 7.6.1. *The spectral surface Σ is isomorphic to the blow-up of $T \times \mathbf{P}^1$ at one point in each of the four fibres of the projection $\pi_1 : T \times \mathbf{P}^1 \to T$ over the points $t_1, \ldots, t_4$.*

The invariance of the Hodge numbers $h^{0,i}$ under blow-ups [32, p.387] then gives:

Lemma 7.6.2. *The spectral surface Σ has the following invariants:*

$$\chi(\Sigma, \mathcal{O}_\Sigma) = 0; \quad h^{0,1}(\Sigma) = 1, \quad \textit{and } h^{0,2}(\Sigma) = 0.$$

We now have the following commutative diagram of double covers and blow-ups:

$$\begin{tikzcd}
\mathbf{P}^1 \times T & \Sigma \arrow[l, "\tilde{\sigma}'"'] \arrow[r, "\tilde{\rho}"] \arrow[d, "\nu"] & T \arrow[d] \\
\mathbf{P}^1 \times \mathbf{P}^1 \arrow[from=1-1, "\tilde{q}"'] & D \arrow[l, "\sigma'"'] \arrow[r, "\rho"] \arrow[d, "\sigma"] & \mathbf{P}^1 \\
& \mathbf{P}^1_0 \times \mathbf{P}^1_\infty &
\end{tikzcd}$$

Denote the pull-back $\sigma^* Z$ by Z_D and $(\sigma \circ \nu)^* Z$ by Z_Σ; then Z_D is the blow-up of Z along the elliptic fibres $\pi^{-1}(b)$ and $\pi^{-1}(\tau(b))$ and there is a natural double cover $\tilde{\nu} : Z_\Sigma \to Z_D$, branched over the divisor $\pi_D^* B$ on Z_D. Let $\pi_\Sigma : Z_\Sigma \to \Sigma$ and $\pi_D : Z_D \to D$ be the obvious elliptic fibrations. Denoting the blow-up $Z_D \to Z$ again by σ, let $\mathcal{E}_D = \sigma^* \mathcal{E}$ and $\mathcal{E}_\Sigma = \tilde{\nu}^* \mathcal{E}_D$. The divisor of $\mathcal{E}_\Sigma$ in $\Sigma \times \mathbf{P}^1$ is tautologically the pull-back of $D \subset Q \times \mathbf{P}^1$ via the map $\sigma \circ \nu$, and its pre-image under $\tilde{q} : \Sigma \times T \to \Sigma \times \mathbf{P}^1$ decomposes as the sum $\gamma + \iota^* \gamma$, where γ is the graph of the tautological map

$$\Sigma \hookrightarrow Q \times T \longrightarrow T.$$

We would now like to apply a 3-dimensional version of the Friedman construction. By (7.6.2), however, we have

$$H^2(\Sigma; \mathcal{O}_\Sigma^*) \cong H^3(\Sigma; \mathbf{Z}) \neq 0,$$

so we cannot use Corollary 2.2.8 to deduce the existence of a line bundle in $\operatorname{Pic}^0(Z, \Sigma)$ with graph γ. We can, however, construct one "by hand".

Lemma 7.6.3. *There exists a line bundle $\mathcal{L}_0$ in* $\operatorname{Pic}^0(Z, \Sigma)$ *with graph* γ.

Proof of 7.6.3: Since Z_D is isomorphic to the quotient of the principal $\mathbf{C}^*$-bundle associated to $\sigma^* \mathcal{O}(1, -1)$ by an action of $\mathbf{Z}$, its restriction to $D \setminus C$ is trivial, where C is the proper transform of any $(1,1)$ curve on Q. In particular, if D_1 and D_2 are the two exceptional fibres of ρ, then $Z_D|(D \setminus (D_1 \cup D_2))$ is trivial and hence so is $Z_\Sigma|(\Sigma \setminus \Lambda)$, where Λ denotes the union of the exceptional fibres $\Sigma_\alpha = \tilde{\rho}^{-1}(t_\alpha)$ $(\alpha = 1, \ldots, 4)$ for $\tilde{\rho}$. The blow up $\tilde{\sigma}' : \Sigma \to \mathbf{P}^1 \times T$ restricts to an isomorphism

$$\tilde{\sigma}' : \Sigma \setminus \Lambda \to \mathbf{P}^1 \times (T \setminus \{t_1, \ldots, t_4\}).$$

The map

$$\gamma \circ (\tilde{\sigma}'|)^{-1} : \mathbf{P}^1 \times (T \setminus \{t_1, \ldots, t_4\}) \to T$$

is then given by $(x,t) \to t$. Hence choosing a trivialisation

$$Z_\Sigma|(\Sigma \setminus \Lambda) \cong \mathbf{P}^1 \times (T \setminus \{t_1,\ldots,t_4\}) \times T,$$

we may define a bundle $\mathcal{L}_0$ over $(\Sigma \setminus \Lambda) \times T$ with the required properties by pulling-back a Poincaré bundle over $\mathrm{Pic}^0(T) \times T$ via the map $(x,t,u) \mapsto (t,u)$.

Now choose disjoint open discs $U_1,\ldots,U_4$ about the points $t_1,\ldots,t_4 \in T$ and let $V_\alpha = \bar{\rho}^{-1}(U_\alpha)$ be the corresponding neighbourhoods of the singular fibres Σ_α. Each V_α deformation retracts onto Σ_α giving $H^3(V_\alpha;\mathbf{Z}) = 0$. Blowing down one of the -1 curves in a singular fibre Σ_α results in a space isomorphic to $U_\alpha \times \mathbf{P}^1$. The invariance of the cohomology of structure sheaves under blow-ups [32, p.387] gives

$$H^2(V_\alpha;\mathcal{O}_{V_\alpha}) \cong H^2(U_\alpha \times \mathbf{P}^1;\mathcal{O}) = 0.$$

Hence, by Corollary 2.2.8, there exists a line bundle $\mathcal{L}_\alpha \in \mathrm{Pic}^0(Z_\Sigma|V_\alpha, V_\alpha)$ whose graph agrees with the restriction of γ.

On each of the intersections

$$V_\alpha \cap (\Sigma \setminus \Lambda) \cong U_\alpha^* \times \mathbf{P}^1,$$

where $U_\alpha^* = U_\alpha \setminus \{t_\alpha\}$, the sheaf $\pi_{D*}(\mathcal{L}_\alpha^{-1}\mathcal{L}_0)$ is a line bundle. If $\pi_2 : U_\alpha^* \times \mathbf{P}^1 \to \mathbf{P}^1$ is projection onto the second factor, then $\pi_{D*}(\mathcal{L}_\alpha^{-1}\mathcal{L}_0) \cong \pi_2^*\mathcal{O}_{\mathbf{P}^1}(n)$ for some $n \in \mathbf{Z}$. Using the blow-up $\sigma_\alpha : V_\alpha \to U_\alpha \times \mathbf{P}^1$, define a line bundle $\mathcal{W}_\alpha$ on V_α by $\mathcal{W}_\alpha = (\pi_2 \circ \sigma_\alpha \circ \pi_\Sigma)^*\mathcal{O}_{\mathbf{P}^1}(n)$, then $\pi_{D*}((\mathcal{L}_\alpha\mathcal{W}_\alpha)^{-1}\mathcal{L}_0)$ is isomorphic to the trivial line bundle on $V_\alpha \cap (\Sigma \setminus \Lambda)$, so by replacing $\mathcal{L}_\alpha$ by $\mathcal{L}_\alpha\mathcal{W}_\alpha$ we may assume there is an isomorphism $\mathcal{L}_\alpha \cong \mathcal{L}_0$ over $Z|(V_\alpha \cap (\Sigma \setminus \Lambda))$. Patching together the $\mathcal{L}_\alpha$ and $\mathcal{L}_0$ via these isomorphisms gives the required bundle.

Lemma 7.6.4. *There exists a line bundle $\mathcal{L} \in \mathrm{Pic}^0(Z_\Sigma, \Sigma)$ such that*

$$\pi_*(\mathcal{L}^{-1}\mathcal{E}_\Sigma) \cong \mathcal{O}_Z.$$

Proof of 7.6.4: Let $\mathcal{L}_0$ be the line bundle supplied by Lemma 7.6.3. Since

$$h^0(\pi_\Sigma^{-1}(x);\mathcal{L}_0^{-1}\mathcal{E}_\Sigma) = 1 \text{ for all } x \in \Sigma,$$

by Grauert's direct image theorem $\mathcal{W} = \pi_{\Sigma*}(\mathcal{L}_0^{-1}\mathcal{E}_\Sigma)$ is a line bundle on Σ. The bundle $\mathcal{L}_0 \otimes \bar{\pi}^*\mathcal{W}$ has the required property. $\boxed{7.6.4}$

Corollary 7.6.5. *The bundle $\mathcal{E}_\Sigma$ fits into a short exact sequence of sheaves:*

7.6.6. $$0 \to \mathcal{L} \to \mathcal{E}_\Sigma \to \mathcal{L}^{-1}\mathcal{J}_Y \to 0,$$

where Y is codimension 2 locally complete intersection subscheme on Z_Σ.

The fibration $\pi_0 : Z \to \mathbf{P}_0^1$ induces corresponding fibrations $\hat{\pi}_0 : Z_D \to \mathbf{P}_0^1$ with generic fibre the Hopf surface X and $\tilde{\pi}_0 : Z_\Sigma \to \mathbf{P}_0^1$ with generic fibre the double cover of the Hopf surface as in Section 5.5. Similarly for π_∞. If

Y' is a component of Y which meets a generic fibre of $\tilde{\pi}_0$ or $\tilde{\pi}_\infty$ transversely, then over $\pi_\Sigma(Y')$ the fibres of $\mathcal{E}_\Sigma$ are of type (3). But any such fibres can lie only over the curves $\nu^* E_i$ $(i = 1, 2)$ on Σ, where the E_i are the exceptional divisors in the blow-up $\sigma : D \to Q$. Hence the components of Y lying outside the exceptional fibres can consist only of a finite number of fibres of π_Σ, and on a generic fibre $\tilde{X}$ of $\tilde{\pi}_0$ or $\tilde{\pi}_\infty$ the exact sequence (7.6.6) restricts to

$$0 \to \mathcal{L}|_{\tilde{X}} \to \mathcal{E}_\Sigma|_{\tilde{X}} \to \mathcal{L}^{-1}|_{\tilde{X}} \to 0.$$

By the results of Chapter 5, the corresponding map

$$(\tilde{\nu}_* \mathcal{L}) \otimes \pi_D^* \mathcal{O}_D(V) \to \mathcal{E}_D$$

is an isomorphism on restriction to the generic fibre of $\hat{\pi}_0$ or $\hat{\pi}_\infty$, where V is the unique divisor class on D such that $B \sim 2V$.

The double cover $\tilde{\nu} : Z_\Sigma \to Z_D$ is the one associated to the square root $\pi_D^* V$ of $\pi_D^* B$ on Z_D, so $\tilde{\nu}_* \mathcal{O}_{Z_\Sigma} \cong \mathcal{O}_{Z_D} \oplus \mathcal{O}_{Z_D}(-\pi_D^* V)$. Pushing down the exact sequence (7.6.6) and using the projection formula results in a non-trivial map of locally free sheaves

7.6.7. $$\Phi : \tilde{\nu}_* \mathcal{L} \to \mathcal{E}_D(-\pi_D^* V),$$

exactly as in Chapter 5.

Lemma 7.6.8. *The map of sheaves (7.6.7) is injective.*

Proof of 7.6.8: Let $\mathcal{K}$ denote the kernel of Φ, so we have an exact sequence

$$0 \to \mathcal{K} \to \tilde{\nu}_* \mathcal{L} \to \mathcal{E}_D(-\pi_D^* V).$$

The sheaf $\mathcal{K}$ is then locally a 2nd syzygy sheaf and hence locally free outside a finite set of points $S \subset Z_D$. Since Φ is non-trivial, the rank of $\mathcal{K}$ is 0 or 1.

Restricting to the set $Z_D \setminus S$ gives an exact sequence of locally free sheaves

$$\mathcal{K} \to \tilde{\nu}_* \mathcal{L} \xrightarrow{\Phi} \mathcal{E}_D(-\pi_D^* V),$$

and, by the remarks preceding the lemma, the map Φ is generically of rank 2. Hence $\mathcal{K}$ has rank 0 and so is a torsion sheaf supported on S. But $\mathcal{K}$ is a sub-sheaf of a torsion free sheaf and hence must be 0. $\boxed{7.6.8}$

As a result of the lemma we have an exact sequence

7.6.9. $$0 \to (\tilde{\nu}_* \mathcal{L}) \otimes \mathcal{O}_{Z_D}(\pi_D^* V) \to \mathcal{E}_D \to \tilde{Q} \to 0,$$

where $\tilde{Q}$ is a torsion sheaf supported on a divisor in Z_D. Let $\mathcal{L}'$ be the line bundle $\mathcal{L} \otimes \mathcal{O}_{Z_\Sigma}(\tilde{\nu}^* \pi_D^* V)$, then (7.6.9) becomes

7.6.10. $$0 \to \tilde{\nu}_* \mathcal{L}' \xrightarrow{\Phi} \mathcal{E}_D \to \tilde{Q} \to 0.$$

On the fibres of Z_D over the exceptional divisor $E_1 \subset D$ the graph of $\tilde{\nu}_*\mathcal{L}'$ is that of a degree 1 rational map from $E_1 \cong \mathbf{P}^1$ to $\mathbf{P}^1$. In particular, the bundle $\tilde{\nu}_*\mathcal{L}'$ is non-trivial on $Z_D|E_1$ and so $\tilde{\nu}_*\mathcal{L}'$ is not the pull-back of a bundle on Z. Nevertheless, pushing down the exact sequence (7.6.10) gives an exact sequence of coherent sheaves

7.6.11. $$0 \to \sigma_*\tilde{\nu}_*\mathcal{L}' \to \mathcal{E} \to \mathcal{Q}_1 \to 0,$$

where $\mathcal{Q}_1$ is a torsion sheaf on Z. Moreover, the coherent rank 2 sheaf $\sigma_*\tilde{\nu}_*\mathcal{L}'$ is naturally identified with the locally free sheaf $\tilde{\nu}_*\mathcal{L}'$ on the complement of the fibres $\pi^{-1}(b), \pi^{-1}(\tau(b))$ in Z where $\mathcal{E}$ is of type (3). Taking the dual of (7.6.11) gives

$$0 \to \mathcal{E} \to (\sigma_*\tilde{\nu}_*\mathcal{L}')^* \to \mathcal{Q}_2 \to 0,$$

for some torsion sheaf $\mathcal{Q}_2$. Taking duals once again, we obtain

7.6.12. $$0 \to (\sigma_*\tilde{\nu}_*\mathcal{L}')^{**} \xrightarrow{\Theta} \mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is yet another torsion sheaf. The sheaf $(\sigma_*\tilde{\nu}_*\mathcal{L}')^{**}$ is reflexive and hence locally free outside a finite set of points $S' \subset E$, where

$$E = \pi^{-1}(b) \cup \pi^{-1}(\tau(b)).$$

This suggests the following result.

Theorem 7.6.13. *The sheaf $(\sigma_*\tilde{\nu}_*\mathcal{L}')^{**}$ is locally free and isomorphic to $\mathcal{E}$.*

Proof of 7.6.13: Let F be the divisor where the torsion sheaf $\tilde{\mathcal{Q}}$ of (7.6.9) is supported, so F is the zero-divisor of the natural map

$$\det(\tilde{\nu}_*\mathcal{L}') \to \det \mathcal{E}_D \cong \mathcal{O}_{Z_D},$$

and is supported in the fibres of π_D. Since the map Φ of (7.6.9) restricts to an isomorphism on generic fibres of $\tilde{\pi}_0$ and $\tilde{\pi}_\infty$, the divisor F must be supported in

$$\tilde{E} = \pi_D^{-1}(E_1) \cup \pi_D^{-1}(E_2),$$

so Φ is an isomorphism on $Z_D \setminus \tilde{E}$. Hence the pushed-down map $\sigma_*\tilde{\nu}_*\mathcal{L}' \to \mathcal{E}$ is an isomorphism outside $E = \pi^{-1}(b) \cup \pi^{-1}(\tau(b))$, from which we deduce that the map $\Theta : (\sigma_*\tilde{\nu}_*\mathcal{L}')^{**} \to \mathcal{E}$ of (7.6.12) is an isomorphism outside E. Hence, on $Z \setminus E$ there is a splitting $\Psi : \mathcal{E} \to (\sigma_*\tilde{\nu}_*\mathcal{L}')^{**}$ of the exact sequence (7.6.12). Since $(\sigma_*\tilde{\nu}_*\mathcal{L}')^{**}$ is reflexive and so normal, the splitting Ψ extends over the codimension 2 subset E giving a global isomorphism

$$(\sigma_*\tilde{\nu}_*\mathcal{L}')^{**} \cong \mathcal{E}.$$

$\boxed{7.6.13}$

The rôle of the blow-up in the proof was crucial. By showing that all the 'bad' points were contained in the exceptional divisor, we were able to blow-down and reduce the dimension of the set of 'bad' points to one which allowed us to perform the 'double-dual' trick, and thereby remove them.

This result suggests a method for the construction of instanton bundles, which if the constraints on the spectral surface for arbitrary instanton number can be determined, ought to generalise also to these. It does not, unfortunately, provide a very useful method for explicitly finding the corresponding connections, but could perhaps be helpful in the study of more general properties of the moduli spaces. In particular, it may be helpful in the study of the action of the isometry group of $S^1 \times S^3$ on the moduli space.

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